Lectures on Differential Equations of Mathematical Physics

A First Course

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NOVA

Lectures on Differential Equations of Mathematical Physics: A First Course

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Preface

The theory of partial differential equations of mathematical physics has been one of the most important fields of study in applied mathematics. This is essentially due to the frequent occurrence of partial differential equations in many branches of natural sciences and engineering.

With much interest and great demand for applications in diverse areas of sciences, several excellent books on differential equations of mathematical physics have been published (see, for example, [1]-[6] and the references therein). The present lecture notes have been written for the purpose of presenting an approach based mainly on the mathematical problems and their related solutions. The primary concern, therefore, is not with the general theory, but to provide students with the fundamental concepts, the underlying principles, and the techniques and methods of solution of partial differential equations of mathematical physics. One of our main goal is to present a fairly elementary and complete introduction to this subject which is suitable for the "first reading" and accessible for students of different specialities.

The material in these lecture notes has been developed and extended from a set of lectures given at Saratov State University and reflects partially the research interests of the authors. It is intended for graduate and advanced undergraduate students in applied mathematics, computer sciences, physics, engineering, and other specialities. The prerequisites for its study are a standard basic course in mathematical analysis or advanced calculus, including elementary ordinary differential equations.

Although various differential equations and problems considered in these lecture notes are physically motivated, a knowledge of the physics involved is not necessary for understanding the mathematical aspects of the solution of these problems.

The book is organized as follows. In Chapter 1 we present the most important examples of equations of mathematical physics, give their classification and discuss formulations of problems of mathematical physics. Chapter 2 is devoted to hyperbolic partial differential equations which usually describe oscillation processes and give a mathematical description of wave propagation. The prototype of the class of hyperbolic equations and one of the most important differential equations of mathematical physics is the wave equation. Hyperbolic equations occur in such diverse fields of study as electromagnetic theory, hydrodynamics, acoustics, elasticity and quantum theory. In this chapter we study hyperbolic equations in one-, two- and three-dimensions, and present methods for their solutions. In Sections 2.1-2.5 we study the main classical problems for hyperbolic equations, namely, the Cauchy problem, the Goursat problem and the mixed problems. We present the main methods for their solutions including the method of travelling waves, the method of separation of variables, the method of successive approximations, the Riemann method, the Kirchhoff method. Thus, Sections 2.1-2.5 contain the basic classical theory for hyperbolic partial differential equations in the form which is suitable for the “first reading”. Sections 2.6-2.9 are devoted to more specific modern problems for differential equations, and can be ommited for the “first reading”. In Sections 2.6-2.8 we provide an elementary introduction to the theory of inverse problems. These inverse problems consist in recovering coefficients of differential equations from characteristics which can be measured. Such problems often appear in various areas of natural sciences and engineering. Inverse problems also play an important role in solving nonlinear evolution equations in mathematical physics such as the Korteweg-de Vries equation. Interest in this subject has been increasing permanently because of the appearance of new important applications, and nowadays the inverse problem theory develops intensively all over the world. In Sections 2.6-2.8 we present the main results and methods on inverse problems and show connections between spectral inverse problems and inverse problems for the wave equation. In Section 2.9 we provide the solution of the Cauchy problem for the nonlinear Korteweg-de Vries equation, for this purpose we use the ideas and results from Sections 2.6-2.8 on the inverse problem theory.

In Chapter 3 we study parabolic partial differential equations which usually describe various diffusion processes. The most important equation of parabolic type is the heat equation or diffusion equation. The properties of the solutions of parabolic equations do not depend essentially on the dimension of the space, and therefore we confine ourselves to considerations concerning the case of one spatial variable. Chapter 4 is devoted to elliptic equations which usually describe stationary fields, for example, gravitational, electrostatical and temperature fields. The most important equations of elliptic type are the Laplace and the Poisson equations. In this chapter we study boundary value problems for elliptic partial differential equations and present methods for their solutions such that the Green’s function method, the method of upper and lower functions, the method of integral equations, the variational method. In Chapter 5 we prove the general Cauchy-Kowalevsky theorem which is a fundamental theorem on the existence of the solution of the Cauchy problem for a wide class of systems of partial differential equations. Chapter 6 contains exercises for the material covered in Chapters 1-4. The material here reflects all main types of equations of mathematical physics and represents the main methods for the solution of these equations.

G. FREILING AND V. YURKO

Chapter 1.

Introduction

1.1. Some Examples of Equations of Mathematical Physics

The theory of partial differential equations arose from investigations of some important problems in physics and mechanics. Most of the problems lead to second-order differential equations which will be (mainly) under our consideration.

The relation

$$\Phi\left(x_1,\ldots,x_n,u,\frac{\partial u}{\partial x_1},\ldots,\frac{\partial u}{\partial x_n},\frac{\partial^2 u}{\partial x_i\partial x_j}\,(i,j=\overline{1,n})\right)=0, \tag{1.1.1}$$

which connects the real variables $x_1,x_2,\ldots,x_n$, an unknown function $u(x_1,\ldots,x_n)$ and its partial derivatives up to the second order, is called a partial differential equation of the second order.

Equation (1.1.1) is called linear with respect to the second derivatives (or quasi-linear) if it has the form

$$\sum_{i,j=1}^{n} a_{ij}(x_1,\ldots,x_n)\frac{\partial^2 u}{\partial x_i\partial x_j}+F\left(x_1,\ldots,x_n,u,\frac{\partial u}{\partial x_1},\ldots,\frac{\partial u}{\partial x_n}\right)=0. \tag{1.1.2}$$

Equation (1.1.2) is called linear if it has the form

$$\sum_{i,j=1}^{n} a_{ij}(x_1,\ldots,x_n)\frac{\partial^2 u}{\partial x_i\partial x_j}+\sum_{i=1}^{n} b_i(x_1,\ldots,x_n)\frac{\partial u}{\partial x_i}+b_0(x_1,\ldots,x_n)u$$

$$=f(x_1,\ldots,x_n). \tag{1.1.3}$$

Let us give several examples of problems of mathematical physics which lead to partial differential equations.

1. Equation of a vibrating string.

Consider a stretched string which lies along the x-axis, and suppose for simplicity that the oscillations take place in a plane, and that all points of the string move perpendicularly to the x-axis. Then the oscillation process can be described by a function $u(x,t)$, which characterizes the vertical displacement of the string at the moment t. Suppose that the string is homogeneous, inextensible, its thickness is constant, and it does not resist bending. We consider only "small" oscillations with $(u_x)^2 << 1$. Then the function $u(x,t)$ satisfies the following equation (see [1, p.23], [2, p.56]):

$$u_{tt} = a^2 u_{xx} + f(x,t), \tag{1.1.4}$$

where $a > 0$ is the wave velocity, and $f(x,t)$ represents a known external force. More general equations for one-dimensional oscillations have the form

$$\rho(x)u_{tt} = (k(x)u_x)_x - q(x)u + f(x,t), \tag{1.1.5}$$

where $\rho(x), k(x)$ and $q(x)$ are specified by properties of the medium. For example, the equation of longitudinal oscillations of a rod has the form (1.1.5), where $u(x,t)$ is the displacement of the point x from the equilibrium at the moment t, $\rho(x)$ is the density of the rod, $k(x)$ is the elasticity coefficient, $q(x) = 0$ and $f(x,t)$ is the density of external forces per unit length [1, p.27].

2. Equation of transverse oscillations of a membrane.

Consider a thin homogeneous inextensible membrane. Suppose that the membrane does not resist bending, and that all its points move perpendicularly to the (x,y)-plane. Let $u(x,y,t)$ be the vertical displacement of the membrane at the moment t. We consider "small" transverse oscillations with $(u_x)^2 + (u_y)^2 << 1$. Then the function $u(x,y,t)$ satisfies the following equation (see [1, p.31]):

$$u_{tt} = a^2(u_{xx} + u_{yy}) + f(x,y,t),$$

where $a > 0$, and $f(x,y,t)$ represents a known external force.

3. Wave equation.

Multidimensional oscillation processes in the space of n spatial variables
$x = (x_1, \ldots, x_n)$ are described by the so-called wave equation

$$u_{tt} = a^2 \Delta u + f(x,t), \tag{1.1.6}$$

where

$$\Delta u := \sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2}$$

is the Laplace operator (or Laplacian), a is the speed of wave propagation, and $f(x,t)$ represents external forces. For $n=1$ equation (1.1.6) coincides with the equation of a vibrating string (1.1.4). Therefore, equation (1.1.4) is also called the one-dimensional wave equation. For $n=2$ equation (1.1.6) describes, for example, two-dimensional oscillations of a membrane [1, p.31]. Propagation of acoustic waves in the space of three spatial variables is described by the wave equation for $n=3$. More general wave equations have the form

$$\rho(x)u_{tt} = \operatorname{div}(k(x)\operatorname{grad}u) - q(x)u + f(x,t), \tag{1.1.7}$$

where

$$\operatorname{div}(k(x)\operatorname{grad}u) := \sum_{k=1}^{n} \frac{\partial}{\partial x_k}\Big(k(x)\frac{\partial u}{\partial x_k}\Big),$$

where the functions $\rho(x), k(x)$ and $q(x)$ describe the properties of the medium.

4. Telegrapher's equation.

An important particular case of the one-dimensional wave equation is the so-called telegrapher's equation. Consider the system of differential equations

$$\left.\begin{aligned} i_x + Cv_t + Gv = 0,\\ v_x + Li_t + Ri = 0, \end{aligned}\right\} \tag{1.1.8}$$

which governs the transmission of signals in telegraph lines. Here v denotes the voltage and i denotes the current; L, C, R and G represent inductance, capacity, resistance and conductivity, respectively (per unit length) [2, p.88]. After elimination of i (or v) from (1.1.8) we arrive at the equation

$$w_{xx} = a_0 w_{tt} + 2b_0 w_t + c_0 w, \tag{1.1.9}$$

where $w = i$ or v (the same equation is obtained for the voltage and the current), and

$$a_0 = LC, \quad 2b_0 = RC + GL, \quad c_0 = GR.$$

Equation (1.1.9) is called the telegrapher's equation. The replacement

$$w = u\exp\left(-\frac{b_0 t}{a_0}\right)$$

reduces (1.1.9) to the simpler form

$$u_{tt} = a^2 u_{xx} + Bu, \tag{1.1.10}$$

where

$$a = \frac{1}{\sqrt{a_0}}, \quad B = \frac{b_0^2 - a_0 c_0}{a_0^2}.$$

We note that $B = 0$ if and only if $RC = GL$ (Heaviside's condition) which corresponds to lines without distortion.

5. Equation of oscillations of a rod.

In a standard course devoted to partial differential equations of mathematical physics we usually deal with second order equations. However, there are a number of problems of mathematical physics which lead to equations of higher order. As an example we consider the problem of free oscillations of a thin rectangular rod when one of its ends is fixed. In this case the oscillation process is described by the following fourth order equation (see [1, p.143]):
$$\frac{\partial^2 u}{\partial t^2}+a^2\frac{\partial^4 u}{\partial x^4}=0.$$
This equation is also used for studying the stability of rotating shafts, vibrations of ships and for other problems of mathematical physics (see [1, Ch.2]).
Many other equations of science and engineering are of order higher than two. In plane elasticity one meets the fourth order biharmonic equation $\nabla^4\, u=0$, in shell analysis eight order equations and in dynamics of three-dimensional travelling strings sixth order equations - below we shall not study higher order equations in detail.

6. Heat equation.

Consider a long slender homogeneous rod of uniform cross sections which lies along the x- axis. The lateral surface of the bar is insulated. The propagation of heat along the bar is described by the equation [1, p.180]

$$u_t=a^2u_{xx}+f(x,t), \tag{1.1.11}$$

which is called the heat conduction equation or, shorter, heat equation. Here the variable t has the significance of time, x is the spatial variable, $u(x,t)$ is the temperature, $a>0$ is a constant which depends on physical properties of the rod, and $f(x,t)$ represents the density of heat sources. Equation (1.1.11) describes also various diffusion processes, therefore it is also called the diffusion equation. A more general one-dimensional heat equation has the form
$$\rho(x)u_t=(k(x)u_x)_x-q(x)u+f(x,t).$$
Propagation of heat in the space of n spatial variables $x=(x_1,\dots,x_n)$ is described by the multidimensional heat conduction equation

$$u_t=a^2\Delta u+f(x,t), \tag{1.1.12}$$

or, in the more general case,

$$\rho(x)u_t=\mathrm{div}(k(x)\mathrm{grad}u)-q(x)u+f(x,t). \tag{1.1.13}$$

7. Diffusion equation.

Consider a medium which is filled non-uniformly by a gas. This leads to a diffusion of the gas from domains with higher concentrations to domains with less ones. Similar processes take place in non-homogeneous chemical solutions.

Consider an one-dimensional diffusion process for a gas in a thin tube which lies along the x-axis, and suppose that the diffusion coefficient is a constant. Denote by $u(x,t)$ the concentration of the gas at the moment t. Then the function $u(x,t)$ satisfies the equation (see [1, p.184]):

$$u_t = a^2 u_{xx}, \quad a > 0.$$

8. Laplace equation.

Let $x = (x_1, \dots, x_n)$ be the spatial variables, and let $u(x)$ be an unknown function. The equation

$$\Delta u = 0 \tag{1.1.14}$$

is called the Laplace equation. Many important physical and mathematical problems are reduced to the solution of the Laplace equation. Usually the Laplace equation describes stationary fields. For example, suppose we study a temperature distribution in some domain of an n-dimensional space of variables $x = (x_1, \dots, x_n)$. In the simplest case the heat propagation is described by the equation $u_t = \Delta u$. A stationary (steady-state) temperature distribution is a temperature field which is independent of time. Such a state may be realized if a sufficiently long time has elapsed from the start of heat flow. Then $u_t = 0$, and the heat equation $u_t = \Delta u$ reduces to the Laplace equation $\Delta u = 0$.

The non-homogeneous Laplace equation

$$\Delta u = f(x) \tag{1.1.15}$$

is called the *Poisson equation*. For example, the gravitational (electro-statical) potential $u(x)$, generated by a body with the mass (charge) density $\rho(x)$, satisfies equation (1.1.15) for $f(x) = -4\pi\rho(x)$. More general partial differential equations, which describe stationary processes (fields), have the form

$$\mathrm{div}(k(x)\mathrm{grad}u) - q(x)u = f(x).$$

Other examples of problems in physics and mechanics, which lead to partial differential equations, and the derivation of the main equations of mathematical physics one can find, for example, in [1]-[3].

1.2. Classification of Second-Order Partial Differential Equations

1. Classification of second-order equations with two variables

We consider quasi-linear second-order partial differential equations with two independent variables x, y of the form:

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + F(x,y,u,u_x,u_y) = 0. \tag{1.2.1}$$

Here $u(x,y)$ is an unknown function and $a_{jk} = a_{jk}(x,y),\ 1 \le j,k \le 2,$ are given continuous functions which are not zero simultaneously. The properties of the solutions of equation (1.2.1) depend mainly on the terms containing second derivatives, i.e. on the coefficients $a_{jk}(x,y)$. Let us give a classification of equations of the form (1.2.1) with respect to $a_{jk}(x,y)$.

Definition 1.2.1.
1) Equation (1.2.1) is called *hyperbolic* if $a_{12}^2 - a_{11}a_{22} > 0$.
2) Equation (1.2.1) is called *parabolic* if $a_{12}^2 - a_{11}a_{22} = 0$.
3) Equation (1.2.1) is called *elliptic* if $a_{12}^2 - a_{11}a_{22} < 0$.

The type of equation (1.2.1) is defined point-wise. If equation (1.2.1) has the same type in all points of a certain domain, then by a suitable change of variables (i.e. by changing the coordinate system) one can achieve that the equation will be in the "simplest" (canonical) form. Let us make the following change of variables:

$$\xi = \varphi(x,y),\ \eta = \psi(x,y), \tag{1.2.2}$$

where φ and ψ are twice continuously differentiable functions, and

$$\begin{vmatrix} \varphi_x & \psi_x \\ \varphi_y & \psi_y \end{vmatrix} \ne 0. \tag{1.2.3}$$

We note that (1.2.3) is the condition for the local invertibility of the transformation (1.2.2). Then

$$u_x = u_\xi \varphi_x + u_\eta \psi_x, \quad u_y = u_\xi \varphi_y + u_\eta \psi_y,$$
$$u_{xx} = u_{\xi\xi}\varphi_x^2 + 2u_{\xi\eta}\varphi_x\psi_x + u_{\eta\eta}\psi_x^2 + u_\xi\varphi_{xx} + u_\eta\psi_{xx},$$
$$u_{xy} = u_{\xi\xi}\varphi_x\varphi_y + u_{\xi\eta}(\varphi_x\psi_y + \varphi_y\psi_x) + u_{\eta\eta}\psi_x\psi_y + u_\xi\varphi_{xy} + u_\eta\psi_{xy},$$
$$u_{yy} = u_{\xi\xi}\varphi_y^2 + 2u_{\xi\eta}\varphi_y\psi_y + u_{\eta\eta}\psi_y^2 + u_\xi\varphi_{yy} + u_\eta\psi_{yy}.$$

Substituting this into (1.2.1) we obtain

$$\tilde a_{11}u_{\xi\xi} + 2\tilde a_{12}u_{\xi\eta} + \tilde a_{22}u_{\eta\eta} + \tilde F(\xi,\eta,u,u_\xi,u_\eta) = 0, \tag{1.2.4}$$

where

$$\left.\begin{aligned} \tilde a_{11} &= a_{11}\varphi_x^2 + 2a_{12}\varphi_x\varphi_y + a_{22}\varphi_y^2, \\ \tilde a_{12} &= a_{11}\varphi_x\psi_x + a_{12}(\varphi_x\psi_y + \psi_x\varphi_y) + a_{22}\varphi_y\psi_y, \\ \tilde a_{22} &= a_{11}\psi_x^2 + 2a_{12}\psi_x\psi_y + a_{22}\psi_y^2. \end{aligned}\right\} \tag{1.2.5}$$

Remark 1.2.1. One can easily check that

$$\tilde{a}_{12}^2 - \tilde{a}_{11}\tilde{a}_{22} = (a_{12}^2 - a_{11}a_{22})\Omega^2,$$

where $\Omega := \varphi_x\psi_y - \psi_x\varphi_y \neq 0$. This means the invariance of the type with respect to the replacement (1.2.2).

We choose the replacement (1.2.2) such that equation (1.2.4) takes the "simplest" (canonical) form. In (1.2.2) we have two arbitrary functions φ and ψ. Let us show that they can be chosen such that one of the following conditions is fulfilled:

1) $\tilde{a}_{11} = \tilde{a}_{22} = 0$;
2) $\tilde{a}_{11} = \tilde{a}_{12} = 0$ (or symmetrically $\tilde{a}_{22} = \tilde{a}_{12} = 0$);
3) $\tilde{a}_{11} = \tilde{a}_{22}, \tilde{a}_{12} = 0$.

Let for definiteness $a_{11} \neq 0$. Otherwise, either $a_{22} \neq 0$ (and then interchanging places of x and y, we get an equation in which $a_{11} \neq 0$), or $a_{11} = a_{22} = 0$, $a_{12} \neq 0$ (and then equation (1.2.1) already has the desired form). So, let $a_{11} \neq 0$. We consider the following ordinary differential equation

$$a_{11}\left(\frac{dy}{dx}\right)^2 - 2a_{12}\frac{dy}{dx} + a_{22} = 0, \tag{1.2.6}$$

which is called the characteristic equation for (1.2.1). The characteristic equation can be also written in the symmetrical form:

$$a_{11}(dy)^2 - 2a_{12}dydx + a_{22}(dx)^2 = 0. \tag{1.2.6$'$}$$

Solutions of the characteristic equation are called the characteristics (or characteristic curves) of (1.2.1). Equation (1.2.6) is equivalent to the following two equations

$$\frac{dy}{dx} = \frac{a_{12} \pm \sqrt{a_{12}^2 - a_{11}a_{22}}}{a_{11}}. \tag{1.2.6$_\pm$}$$

According to Definition 1.2.1, the sign of the expression $a_{12}^2 - a_{11}a_{22}$ defines the type of equation (1.2.1).

Lemma 1.2.1. *If $\varphi(x,y) = C$ is a general integral (or solution surface) of equation (1.2.6), then*

$$a_{11}\varphi_x^2 + 2a_{12}\varphi_x\varphi_y + a_{22}\varphi_y^2 = 0. \tag{1.2.7}$$

Proof. Take a point (x_0, y_0) from our domain. For definiteness let $\varphi_y(x_0, y_0) \neq 0$, and let $\varphi(x_0, y_0) = C_0$. Consider the integral curve $y = y(x, C_0)$ of equation (1.2.6) passing through the point (x_0, y_0). Clearly, $y(x_0, C_0) = y_0$. Since $\varphi(x,y) = C$ is a general integral, we have on the integral curve:

$$\varphi_x + \varphi_y\frac{dy}{dx} = 0;$$

hence,

$$\frac{dy}{dx} = -\frac{\varphi_x}{\varphi_y}.$$

Then

$$0 = a_{11}\left(\frac{dy}{dx}\right)^2 - 2a_{12}\frac{dy}{dx} + a_{22}$$

$$= \left(a_{11}\left(\frac{\varphi_x}{\varphi_y}\right)^2 + 2a_{12}\frac{\varphi_x}{\varphi_y} + a_{22}\right)\Bigg|_{y=y(x,C_0)}.$$

In particular, taking $x = x_0$, we get (1.2.7) at the point (x_0, y_0). By virtue of the arbitrariness of (x_0, y_0), the lemma is proved. □

Case 1: $a_{12}^2 - a_{11}a_{22} > 0$ *- hyperbolic type.*
In this case there are two different characteristics of equation (1.2.6) passing through each point of the domain. Let $\varphi(x,y) = C$ and $\psi(x,y) = C$ be the general integrals of equations ($1.2.6_{\pm}$), respectively. These integrals give us two families of characteristics of equation (1.2.6). We note that (1.2.3) holds, since

$$\frac{\varphi_x}{\varphi_y} = -\frac{a_{12} + \sqrt{a_{12}^2 - a_{11}a_{22}}}{a_{11}}, \quad \frac{\psi_x}{\psi_y} = -\frac{a_{12} - \sqrt{a_{12}^2 - a_{11}a_{22}}}{a_{11}},$$

and consequently,

$$\frac{\varphi_x}{\varphi_y} \neq \frac{\psi_x}{\psi_y}.$$

In (1.2.1) we make the substitution (1.2.2), where φ and ψ are taken from the general integrals of equations ($1.2.6_{\pm}$), respectively. It follows from (1.2.5) and Lemma 1.2.1 that $\tilde{a}_{11} = \tilde{a}_{22} = 0$. Therefore, equation (1.2.4) is reduced to the form

$$u_{\xi\eta} = \Phi(\xi, \eta, u, u_\xi, u_\eta), \tag{1.2.8}$$

where $\Phi = -\dfrac{\tilde{F}}{2\tilde{a}_{12}}$. This is the canonical form of equation (1.2.1) for the hyperbolic case. There also exists another canonical form of hyperbolic equations. Put

$$\xi = \alpha + \beta, \quad \eta = \alpha - \beta,$$

i.e.

$$\alpha = \frac{\xi + \eta}{2}, \quad \beta = \frac{\xi - \eta}{2}.$$

Then

$$u_\xi = \frac{u_\alpha + u_\beta}{2}, \quad u_\eta = \frac{u_\alpha - u_\beta}{2}, \quad u_{\xi\eta} = \frac{u_{\alpha\alpha} - u_{\beta\beta}}{4},$$

and consequently, equation (1.2.8) takes the form

$$u_{\alpha\alpha} - u_{\beta\beta} = \Phi_1(\alpha, \beta, u, u_\alpha, u_\beta). \tag{1.2.9}$$

This is the second canonical form of hyperbolic equations. The simplest examples of an hyperbolic equation are the equation of a vibrating string $u_{tt} = u_{xx}$ and the more general equation (1.1.4).

Case 2: $a_{12}^2 - a_{11}a_{22} = 0$ - *parabolic type.*
In this case equations (6_+) and (6_-) coincide, and we have only one general integral of equation (1.2.6): $\varphi(x,y) = C$. We make the substitution (1.2.2), where $\varphi(x,y)$ is a general integral of equation (1.2.6), and $\psi(x,y)$ is an arbitrary smooth function satisfying (1.2.3). By Lemma 1.2.1, $\tilde{a}_{11} = 0$. Furthermore, since $a_{12} = \sqrt{a_{11}}\sqrt{a_{22}}$, we have according to (1.2.5),

$$\tilde{a}_{11} = \left(\sqrt{a_{11}}\varphi_x + \sqrt{a_{22}}\varphi_y\right)^2,$$

and consequently,

$$\sqrt{a_{11}}\varphi_x + \sqrt{a_{22}}\varphi_y = 0.$$

Then

$$\tilde{a}_{12} = \left(\sqrt{a_{11}}\varphi_x + \sqrt{a_{22}}\varphi_y\right)\left(\sqrt{a_{11}}\psi_x + \sqrt{a_{22}}\psi_y\right) = 0,$$

and equation (1.2.1) is reduced to the canonical form

$$u_{\eta\eta} = \Phi(\xi, \eta, u, u_\xi, u_\eta), \tag{1.2.10}$$

where $\Phi = -\dfrac{\tilde{F}}{2\tilde{a}_{22}}$. The simplest example of a parabolic type equation is the heat equation $u_t = u_{xx}$.

Case 3: $a_{12}^2 - a_{11}a_{22} < 0$ - *elliptic type.*
In this case the right-hand sides in ($1.2.6_\pm$) are complex. Let $\varphi(x,y) = C$ be the complex general integral of equation ($1.2.6_+$). Then $\overline{\varphi(x,y)} = C$ is the general integral of equation ($1.2.6_-$). We take the complex variables $\xi = \varphi(x,y)$, $\eta = \overline{\varphi(x,y)}$, i.e. make the substitution (1.2.2), where $\psi = \bar{\varphi}$. Repeating the arguments from Case 1, we deduce that equation (1.2.1) is reduced to the form $u_{\xi\eta} = \Phi(\xi, \eta, u, u_\xi, u_\eta)$. But this equation is not canonical for elliptic equations since ξ, η are complex variables. We pass on to the real variables (α, β) by the replacement $\xi = \alpha + i\beta$, $\eta = \alpha - i\beta$. Then $\alpha = (\xi + \eta)/2$, $\beta = (\xi - \eta)/(2i)$. We calculate $u_\xi = (u_\alpha - iu_\beta)/2$, $u_\eta = (u_\alpha + iu_\beta)/2$, $u_{\xi\eta} = (u_{\alpha\alpha} + u_{\beta\beta})/4$. Therefore, equation (1.2.1) is reduced to the canonical form

$$u_{\alpha\alpha} + u_{\beta\beta} = \Phi_1(\alpha, \beta, u, u_\alpha, u_\beta). \tag{1.2.11}$$

The simplest example of an elliptic equation is the two-dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Remark 1.2.2. Since we used complex variables, our arguments for Case 3 are valid only if $a_{ij}(x,y)$ are analytic functions. In the general case, the reduction to the canonical form for elliptic equations is a more complicated problem. For simplicity, we confined ourselves here to the case of analytic coefficients.

2. Classification of second-order equations with an arbitrary number of variables.

We consider the equation

$$\sum_{i,j=1}^{n} a_{ij}(x_1,\dots,x_n)\frac{\partial^2 u}{\partial x_i \partial x_j} + F\left(x_1,\dots,x_n,u,\frac{\partial u}{\partial x_1},\dots,\frac{\partial u}{\partial x_n}\right) = 0. \tag{1.2.12}$$

Without loss of generality, let $a_{ij} = a_{ji}$. We make the substitution

$$\xi_k = \xi_k(x_1,\dots,x_n), \quad k = \overline{1,n}. \tag{1.2.13}$$

Denote $\alpha_{ik} := \dfrac{\partial \xi_k}{\partial x_i}$. Then

$$\left.\begin{aligned} \frac{\partial u}{\partial x_i} &= \sum_{k=1}^{n} \frac{\partial u}{\partial \xi_k}\alpha_{ik}, \\ \frac{\partial^2 u}{\partial x_i \partial x_j} &= \sum_{k,l=1}^{n} \frac{\partial^2 u}{\partial \xi_k \partial \xi_l}\alpha_{ik}\alpha_{jl} + \sum_{k=1}^{n} \frac{\partial u}{\partial \xi_k}\frac{\partial^2 \xi_k}{\partial x_i \partial x_j}. \end{aligned}\right\} \tag{1.2.14}$$

Substituting (1.2.14) into (1.2.12) we get

$$\sum_{k,l=1}^{n} \tilde{a}_{kl}(\xi_1,\dots,\xi_n)\frac{\partial^2 u}{\partial \xi_k \partial \xi_l} + \tilde{F}\left(\xi_1,\dots,\xi_n,u,\frac{\partial u}{\partial \xi_1},\dots,\frac{\partial u}{\partial \xi_n}\right) = 0,$$

where

$$\tilde{a}_{kl} = \sum_{i,j=1}^{n} a_{ij}\alpha_{ik}\alpha_{jl}.$$

Consider the quadratic form

$$\sum_{i,j=1}^{n} a_{ij}^0 p_i p_j, \tag{1.2.15}$$

where $a_{ij}^0 = a_{ij}(x_1^0,\dots,x_n^0)$ at a fixed point $(x_1^0,\dots,x_n^0)$. The replacement

$$p_i = \sum_{k=1}^{n} \alpha_{ik} q_k$$

reduces (1.2.15) to the form

$$\sum_{k,l=1}^{n} \tilde{a}_{kl}^0 q_k q_l, \quad \text{where} \quad \tilde{a}_{kl}^0 = \sum_{i,j=1}^{n} a_{ij}^0 \alpha_{ik}\alpha_{jl}.$$

Thus, the coefficients of the main part of the differential equation are changed similarly to the coefficients of the quadratic form. It is known from linear algebra (see [10]) that the quadratic form (1.2.15) can be reduced to the diagonal form:

$$\sum_{i,j=1}^{n} a_{ij}^0 p_i p_j = \sum_{l=1}^{m} (\pm q_l^2), \quad m \le n,$$

and the numbers of positive, negative and zero coefficients are invariant with respect to linear transformations.

Definition 1.2.2. 1) If $m = n$ and all coefficients for q_l^2 have the same sign, then equation (1.2.12) is called *elliptic* at the point x_0.
2) If $m = n$ and all signs except one are the same, then equation (1.2.12) is called *hyperbolic* at the point x_0.
3) If $m = n$ and there are more than one of each sign " $+$ " and " $-$ ", then equation (1.2.12) is called *ultrahyperbolic* at the point x_0.
4) If $m < n$, the equation is called *parabolic* at the point x_0.

For example, the Laplace equation $\Delta u = 0$ is elliptic, the wave equation $u_{tt} = \Delta u$ is hyperbolic, and the heat conduction equation $u_t = \Delta u$ is parabolic in the whole space.

1.3. Formulation of Problems of Mathematical Physics

From the theory of ordinary differential equations it is known that solutions of an ordinary differential equation are determined non-uniquely. For example, the general solution of an n -th order equation

$$y^{(n)} = F(x, y, y', \ldots, y^{(n-1)})$$

depends on n arbitrary constants:

$$y = \varphi(x, C_1, \ldots, C_n).$$

In order to determine the unique solution of the equation we need additional conditions, for example, the initial conditions

$$y_{|x=x_0} = \alpha_0, \ldots, y^{(n-1)}_{|x=x_0} = \alpha_{n-1}.$$

For partial differential equations solutions are also determined non-uniquely, and the general solution involves arbitrary functions. Consider, for example, the case of two independent variables (x, y) :

1) the equation

$$\frac{\partial u}{\partial y} = 0$$

has the general solution

$$u = f(x),$$

where $f(x)$ is an arbitrary function of x ;

2) the equation

$$\frac{\partial^2 u}{\partial x \partial y} = 0$$

has the general solution

$$u = f(x) + g(y),$$

where $f(x)$ and $g(y)$ are arbitrary smooth functions of x and y, respectively.

In order to determine a solution of a partial differential equation uniquely it is necessary to specify additional conditions on the unknown function. Therefore, each problem of mathematical physics is formulated as a problem of the solution of a partial differential equation under some additional conditions which are dictated by its physical statement. Let us give several examples of the statement of problems of mathematical physics.

I. Consider the problem of a vibrating string of the length l $(0 < x < l)$, fixed at the ends $x = 0$ and $x = l$. In this case one needs to seek the solution of the equation

$$u_{tt} = a^2 u_{xx} + f(x,t) \tag{1.3.1}$$

in the domain $0 < x < l, t > 0$ under the conditions

$$u_{|x=0} = u_{|x=l} = 0. \tag{1.3.2}$$

Such conditions are called boundary conditions. However, the specification of the behavior of the string at its ends is not sufficient for the description of the oscillation process. It is necessary to specify the position of the string and its speed in the initial moment of time $t = 0$:

$$u_{|t=0} = \varphi(x), \quad u_{t|t=0} = \psi(x). \tag{1.3.3}$$

Thus, we arrive at the problem (1.3.1)-(1.3.3) which is called the *mixed problem for the equation of of the vibrating string*. We note that instead of (1.3.2) one can specify other types of boundary conditions. For example, if the ends of the string are moving according to known rules, then the boundary conditions have the form

$$u_{|x=0} = \mu(t), \quad u_{|x=l} = \nu(t),$$

where $\mu(t)$ and $\nu(t)$ are known functions. If, for example, the end $x = 0$ is freely moved, then the boundary condition at the point $x = 0$ can be written as

$$u_{x|x=0} = 0.$$

If the end $x = 0$ is elastically fixed, then the boundary condition has the form

$$(u_x + hu)_{|x=0} = 0.$$

There are other types of boundary conditions.

II. Consider the vibrations of an infinite string ($-\infty < x < \infty$). This is a model of the case when the length of the string is sufficiently great so that the effect of the ends can be neglected. In this case boundary conditions are absent, and we arrive at the following problem:
Find the solution $u(x,t)$ of equation (1.3.1) in the domain $-\infty < x < \infty, t > 0$, satisfying the initial conditions (1.3.3). This problem is called the *Cauchy problem for the equation of the vibrating string*.

Similarly one can formulate problems in the multidimensional case and also for parabolic partial differential equations (see below Chapters 2-3 for more details).

III. Let us give an example of the statement of problems for elliptic equations. Elliptic equations usually describe stationary fields. For example, the problem of finding the stationary (steady-state) temperature distribution in a domain $D \in \mathbf{R}^3$ of spatial variables $x = (x_1, x_2, x_3)$ under the condition that the fixed temperature $\varphi(x)$ is given on the boundary Σ of D, can be written in the form

$$\Delta u = 0, \quad u_{|\Sigma} = \varphi, \qquad \Delta u := \sum_{k=1}^{3} \frac{\partial^2 u}{\partial x_k^2}. \tag{1.3.4}$$

Problem (1.3.4) is called the *Dirichlet problem.* For other formulations of problems of mathematical physics see [1]-[3].

Well-posed problems

Problems of mathematical physics are mathematical models of physical problems. The solution of the corresponding problem depends, as shown above, on some functions appearing in the equation and the initial and boundary conditions, which are called prescribed (input) data. In the investigations of problems of mathematical physics the central role is played by the following questions:

1) the existence of the solution;
2) its uniqueness;
3) the dependence of the solution on "small" perturbations of the prescribed data.

If "small" perturbations of the prescribed data lead to "small" perturbations of the solution, then we shall say that the solution is *stable*. Of course, in each concrete situation the notion of "small" perturbations should be exactly defined. For example, for the Dirichlet problem (1.3.4), the solution is called stable if for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that if $|\varphi(x) - \tilde{\varphi}(x)| \leq \delta$ for all $x \in \Sigma$, then $|u(x) - \tilde{u}(x)| \leq \varepsilon$ for all $x \in D$ (here u and $\tilde{u}$ are the solutions of the Dirichlet problems with boundary values φ and $\tilde{\varphi}$, respectively).

An important class of problems of mathematical physics, which was introduced by Hadamard, is the class of well-posed problems.

Definition 1.3.1. A problem of mathematical physics is called *well-posed* (or correctly set) if its solution exists, is unique and stable.

In the next chapters we study the questions of stability and well-posedness of problems for each type of equations separately using their specific character. For advanced studies on the theory of well-posed and ill-posed problems we refer to [3] and the references therein.

Hadamard's example

Let us give an example of an *ill-posed* problem, which is due to Hadamard. Consider the Cauchy problem for the Laplace equation:

$$\left.\begin{array}{c} u_{xx} + u_{yy} = 0, \quad -\infty < x < \infty, \quad y > 0, \\ u_{|y=0} = 0, \quad u_{y|y=0} = 0. \end{array}\right\} \tag{1.3.5}$$

Problem (1.3.5) has the unique solution $u \equiv 0$. Together with (1.3.5) we consider the problem

$$\left.\begin{aligned} u_{xx} + u_{yy} = 0, \quad -\infty < x < \infty, \quad y > 0, \\ u_{|y=0} = 0, \quad u_{y|y=0} = e^{-\sqrt{n}} \sin nx, \quad n \geq 1. \end{aligned}\right\} \tag{1.3.6}$$

The solution of problem (1.3.6) has the form

$$u_n(x,y) = \frac{1}{n} e^{-\sqrt{n}} \sin nx \sinh ny,$$

where

$$\sinh ny := \frac{1}{2}\left(e^{ny} - e^{-ny}\right).$$

Since $|\sin nx| \leq 1$, one gets

$$\max_{-\infty<x<\infty} |e^{-\sqrt{n}} \sin nx| \to 0$$

as $n \to \infty$. However, the solution $u_n(x,y)$ increases infinitely. Indeed,

$$u_n(x,y) = \frac{1}{2n} e^{ny-\sqrt{n}} \sin nx (1 - e^{-2ny}).$$

For each fixed $y > 0$,

$$\max_{-\infty<x<\infty} |u_n(x,y)| \to \infty$$

as $n \to \infty$.

Chapter 2.

Hyperbolic Partial Differential Equations

Hyperbolic partial differential equations usually describe oscillation processes and give a mathematical description of wave propagation. The prototype of the class of hyperbolic equations and one of the most important differential equations of mathematical physics is the wave equation (see Section 1.1). Hyperbolic equations occur in such diverse fields of study as electromagnetic theory, hydrodynamics, acoustics, elasticity and quantum theory. In this chapter we study hyperbolic equations in one-, two- and three-dimensions, and present methods for their solutions. We note that in Chapter 6, Section 6.2 one can find exercises, illustrative example and physical motivations for problems related to hyperbolic partial differential equations.

2.1. The Cauchy Problem for the Equation of the Vibrating String

1. Homogeneous equation of the vibrating string

We consider the following Cauchy problem

$$u_{tt} = a^2 u_{xx}, \quad -\infty < x < \infty,\ t > 0, \tag{2.1.1}$$

$$u_{|t=0} = \varphi(x), \quad u_{t|t=0} = \psi(x). \tag{2.1.2}$$

Here x and t are independent variables (x is the spatial variable, and t has the significance of time), $u(x,t)$ is an unknown function, and $a > 0$ is a constant (wave velocity). The Cauchy problem (2.1.1)-(2.1.2) describes the oscillation process for the infinite string provided that the initial displacement $\varphi(x)$ and the initial speed $\psi(x)$ are known. We assume that $\psi(x)$ is continuously differentiable ($\psi \in C^1$), and $\varphi(x)$ is twice continuously differentiable ($\varphi \in C^2$). Denote

$$D = \{(x,t):\ t \geq 0,\ -\infty < x < \infty\}.$$

Definition 2.1.1. A function $u(x,t)$ is called a solution of the Cauchy problem (2.1.1)-(2.1.2), if $u(x,t) \in C^2(D)$ and $u(x,t)$ satisfies (2.1.1) and (2.1.2). Here and below, the notation $u \in C^m(D)$ means that the function u has in D continuous partial derivatives up to the m-th order.

This notion of the solution is called usually *classical solution*. We mention that in the literature one can find other solution concepts like weak solutions, viscosity solutions, etc. (see [3]); such solution concepts are out of the score of this introductory book.

First we construct the general solution of equation (2.1.1). The characteristic equation for (2.1.1) has the form

$$\frac{dx}{dt} = \pm a$$

(see Section 1.2), and consequently, equation (2.1.1) has two families of characteristics:

$$x \pm at = const.$$

The replacement

$$\xi = x + at, \quad \eta = x - at,$$

reduces (2.1.1) to the form

$$u_{\xi\eta} = 0.$$

Since the general solution of this equation is

$$u = f(\xi) + g(\eta),$$

the general solution of equation (2.1.1) has the form

$$u(x,t) = f(x+at) + g(x-at), \tag{2.1.3}$$

where f and g are arbitrary twice continuously differentiable functions.

We will seek the solution of the Cauchy problem (2.1.1)-(2.1.2) using the general solution (2.1.3). Suppose that the solution of problem (2.1.1)-(2.1.2) exists. Then it has the form (2.1.3) with some f and g. Taking the initial conditions (2.1.2) into account we obtain

$$\left. \begin{aligned} f(x) + g(x) &= \varphi(x), \\ af'(x) - ag'(x) &= \psi(x) \end{aligned} \right\}.$$

Integrating the second relation we get

$$\left. \begin{aligned} f(x) + g(x) &= \varphi(x), \\ f(x) - g(x) &= \frac{1}{a}\int_{x_0}^{x} \psi(s)\,ds \end{aligned} \right\},$$

where x_0 is arbitrary and fixed. Solving this system with respect to f and g, we calculate

$$f(x) = \frac{1}{2}\varphi(x) + \frac{1}{2a}\int_{x_0}^{x} \psi(s)\,ds,$$

$$g(x) = \frac{1}{2}\varphi(x) - \frac{1}{2a}\int_{x_0}^{x}\psi(s)\,ds.$$

Substituting this expressions into (2.1.3) we arrive at the formula

$$u(x,t) = \frac{1}{2}\left(\varphi(x+at)+\varphi(x-at)\right) + \frac{1}{2a}\int_{x-at}^{x+at}\psi(s)\,ds, \tag{2.1.4}$$

which is called *D'Alembert's formula*. Thus, we have proved that if the solution of the Cauchy problem (2.1.1)-(2.1.2) exists, then it is given by (2.1.4). From this we get the uniqueness of the solution of problem (2.1.1)-(2.1.2). On the other hand, it is easy to check by direct verification that the function $u(x,t)$, defined by (2.1.4), is a solution of problem (2.1.1)-(2.1.2). Indeed, differentiating (2.1.4) twice with respect to t we obtain

$$u_t(x,t) = \frac{a}{2}\Big(\varphi'(x+at)-\varphi'(x-at)\Big) + \frac{1}{2}\Big(\psi(x+at)+\psi(x-at)\Big),$$

$$u_{tt}(x,t) = \frac{a^2}{2}\Big(\varphi''(x+at)+\varphi''(x-at)\Big) + \frac{a}{2}\Big(\psi'(x+at)-\psi'(x-at)\Big).$$

Analogously, differentiating (2.1.4) twice with respect to x we get

$$u_{xx}(x,t) = \frac{1}{2}\Big(\varphi''(x+at)+\varphi''(x-at)\Big) + \frac{1}{2a}\Big(\psi'(x+at)-\psi'(x-at)\Big).$$

Hence, the function $u(x,t)$ is the solution of problem (2.1.1)-(2.1.2). Thus, we have proved the following assertion.

Theorem 2.1.1. *Let* $\varphi(x) \in C^2$, $\psi(x) \in C^1$. *Then the solution of the Cauchy problem* $(2.1.1)-(2.1.2)$ *exists, is unique and is given by D'Alembert's formula* $(2.1.4)$.

Let us now study the stability of the solution of problem (2.1.1)-(2.1.2).

Definition 2.1.2. The solution of problem (2.1.1)-(2.1.2) is called stable if for any $\varepsilon > 0$, $T > 0$ there exists $\delta = \delta(\varepsilon, T) > 0$ such that if

$$|\varphi(x) - \tilde{\varphi}(x)| \le \delta, \quad |\psi(x) - \tilde{\psi}(x)| \le \delta$$

for all x, then

$$|u(x,t) - \tilde{u}(x,t)| \le \varepsilon$$

for all $(x,t) \in D$. Here $\tilde{u}(x,t)$ is the solution of the Cauchy problem with the initial data $\tilde{\varphi}$, $\tilde{\psi}$.

Let us show that the solution of the Cauchy problem (2.1.1)-(2.1.2) is stable. Indeed, by virtue of (2.1.4) we have

$$|u(x,t) - \tilde{u}(x,t)| \le$$
$$\frac{1}{2}|\varphi(x+at) - \tilde{\varphi}(x+at)| + \frac{1}{2}|\varphi(x-at) - \tilde{\varphi}(x-at)|$$
$$+\frac{1}{2a}\int_{x-at}^{x+at}|\psi(s) - \tilde{\psi}(s)|\,ds.$$

For $\varepsilon > 0$, $T > 0$ choose $\delta = \dfrac{\varepsilon}{1+T}$ and suppose that

$$|\varphi(x) - \tilde{\varphi}(x)| \le \delta, \quad |\psi(x) - \tilde{\psi}(x)| \le \delta$$

for all x. Then it follows from (2.1.5) that

$$|u(x,t) - \tilde{u}(x,t)| \le \frac{\delta}{2} + \frac{\delta}{2} + \frac{\delta}{2a} 2aT = (1+T)\delta = \varepsilon.$$

Thus, the Cauchy problem (2.1.1)-(2.1.2) is well-posed.

Below we will use the following assertion.

Lemma 2.1.1. 1) *If* $\varphi(x)$ *and* $\psi(x)$ *are odd functions, then*

$$u(0,t) = 0.$$

2) *If* $\varphi(x)$ *and* $\psi(x)$ *are even functions, then*

$$u_x(0,t) = 0.$$

Proof. Let

$$\varphi(x) = -\varphi(-x), \quad \psi(x) = -\psi(-x).$$

Then it follows from (2.1.4) for $x = 0$ that

$$u(0,t) = \frac{1}{2}(\varphi(at) + \varphi(-at)) + \frac{1}{2a}\int_{-at}^{at} \psi(s)\,ds = 0.$$

Analogously one can prove the second assertion of the lemma.

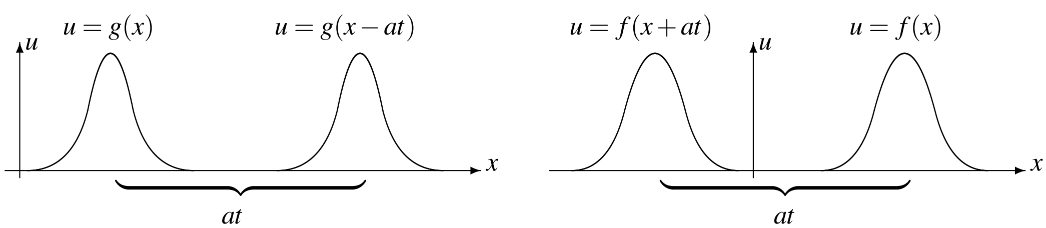

Figure 2.1.1.

Remark 2.1.1. The solutions of the form $u = f(x+at)$ and $u = g(x-at)$ are called *travelling waves*, and a is the wave velocity. The wave $u = f(x+at)$ travels to the left, and the wave $u = g(x-at)$ travels to the right (see fig. 2.1.1). Thus, the general solution (2.1.3) of equation (2.1.1) is the superposition of two travelling waves moving in opposite directions along the x-axis with a common speed.

2. Non-homogeneous equation of the vibrating string

Consider the Cauchy problem for the *non-homogeneous* equation of the vibrating string:

$$\left.\begin{aligned} u_{tt} = a^2 u_{xx} + f(x,t), \quad -\infty < x < \infty,\ t > 0, \\ u_{|t=0} = \varphi(x), \quad u_{t|t=0} = \psi(x). \end{aligned}\right\} \tag{2.1.6}$$

We seek a solution of problem (2.1.6) in the form

$$u(x,t) = \tilde{u}(x,t) + v(x,t),$$

where $\tilde{u}(x,t)$ is the solution of the Cauchy problem for the homogeneous equation

$$\left.\begin{aligned} \tilde{u}_{tt} = a^2 \tilde{u}_{xx}, \quad -\infty < x < \infty,\ t > 0, \\ \tilde{u}_{|t=0} = \varphi(x), \quad \tilde{u}_{t|t=0} = \psi(x), \end{aligned}\right\} \tag{2.1.7}$$

which can be solved by d'Alembert's formula, and $v(x,t)$ is the solution of the problem for the non-homogeneous equation with *zero* initial conditions:

$$\left.\begin{aligned} v_{tt} = a^2 v_{xx} + f(x,t), \quad -\infty < x < \infty,\ t > 0, \\ v_{|t=0} = 0, \quad v_{t|t=0} = 0. \end{aligned}\right\} \tag{2.1.8}$$

It is easy to see that the solution of problem (2.1.8) has the form

$$v(x,t) = \int_0^t w(x,t,\tau)\, d\tau, \tag{2.1.9}$$

where the function $w(x,t,\tau)$ for each fixed $\tau \geq 0$ is the solution of the following auxiliary problem:

$$\left.\begin{aligned} w_{tt} = a^2 w_{xx}, \quad -\infty < x < \infty,\ t > \tau, \\ w_{|t=\tau} = 0,\ w_{t|t=\tau} = f(x,\tau). \end{aligned}\right\} \tag{2.1.10}$$

Indeed, differentiating (2.1.9) with respect to t, we obtain

$$v_t(x,t) = \int_0^t w_t(x,t,\tau)\, d\tau + w(x,t,t).$$

By virtue of (2.1.10) $w(x,t,t) = 0$, and consequently,

$$v_t(x,t) = \int_0^t w_t(x,t,\tau)\, d\tau.$$

Analogously we calculate

$$v_{tt}(x,t) = \int_0^t w_{tt}(x,t,\tau)\, d\tau + f(x,t).$$

Since

$$v_{xx}(x,t) = \int_0^t w_{xx}(x,t,\tau)\, d\tau,$$

and $w_{tt} = a^2 w_{xx}$, we conclude that $v(x,t)$ is a solution of problem (2.1.8).

The solution of (2.1.10) is given by the D'Alembert formula:

$$w(x,t,\tau) = \frac{1}{2a}\int_{x-a(t-\tau)}^{x+a(t-\tau)} f(s,\tau)\,ds.$$

Since the solution of (2.1.7) is also given by the D'Alembert formula, we arrive at the following assertion.

Theorem 2.1.2. *Let* $f(x,t) \in C(D)$ *have a continuous partial derivative* $f_x(x,t)$*, and let* φ *and* ψ *satisfy the conditions of Theorem* 2.1.1*. Then the solution of the Cauchy problem* (2.1.6) *exists, is unique and is given by the formula*

$$u(x,t) = \frac{1}{2}\left(\varphi(x+at)+\varphi(x-at)\right) + \frac{1}{2a}\int_{x-at}^{x+at}\psi(s)\,ds$$
$$+\frac{1}{2a}\int_0^t d\tau \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(s,\tau)\,ds. \tag{2.1.11}$$

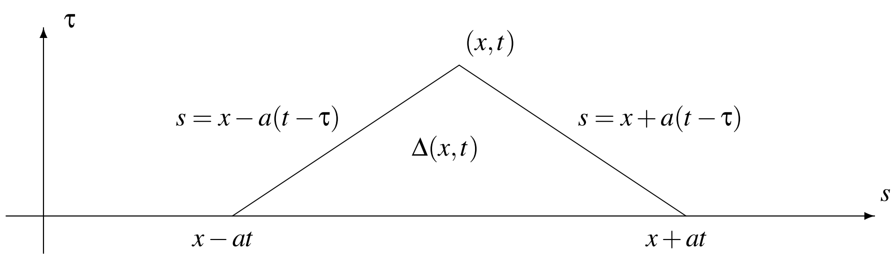

Figure 2.1.2.

Remark 2.1.2. It follows from (2.1.11) that the solution $u(x,t)$ at a fixed point (x,t) depends on the prescribed data f,φ and ψ only from the characteristic triangular $\Delta(x,t) = \{(s,\tau) : x-a(t-\tau) \le s \le x+a(t-\tau), 0 \le \tau \le t\}$ (see fig. 2.1.2). In other words, changing the functions f,φ and ψ outside $\Delta(x,t)$ does not change $u(x,t)$ at the given point (x,t). Physically this means that the wave speed is finite.

3. The semi-infinite string

Consider now the *problem for the semi-infinite string*:

$$\left.\begin{array}{c} u_{tt} = a^2 u_{xx}, \quad x>0,\ t>0, \\ u_{|x=0} = 0, \\ u_{|t=0} = \varphi(x), \quad u_{t|t=0} = \psi(x). \end{array}\right\} \tag{2.1.12}$$

In order to solve this problem we apply the so-called *reflection method*. Let $\psi(x) \in C^1$, $\varphi(x) \in C^2$, $\varphi(0) = \varphi''(0) = \psi(0) = 0$. We extend $\varphi(x)$ and $\psi(x)$ to the whole axis $-\infty < x < \infty$ as odd functions, i.e. we consider the functions

$$\Phi(x) = \begin{cases} \varphi(x), & x \ge 0, \\ -\varphi(-x), & x < 0, \end{cases}$$

$$\Psi(x) = \begin{cases} \psi(x), & x \geq 0, \\ -\psi(-x), & x < 0. \end{cases}$$

According to the D'Alembert formula the function

$$u(x,t) = \frac{1}{2}\left(\Phi(x+at) + \Phi(x-at)\right) + \frac{1}{2a}\int_{x-at}^{x+at} \Psi(s)\, ds \tag{2.1.13}$$

is the solution of the Cauchy problem

$$u_{tt} = a^2 u_{xx}, \quad -\infty < x < \infty,\ t > 0;$$

$$u_{|t=0} = \Phi(x), \quad u_{t|t=0} = \Psi(x).$$

By Lemma 2.1.1 we have $u_{|x=0} = 0$. We consider the function (2.1.13) for $x \geq 0, t \geq 0$. Then

$$u_{|t=0} = \Phi(x) = \varphi(x), \quad x \geq 0,$$

$$u_{t|t=0} = \Psi(x) = \psi(x), \quad x \geq 0.$$

Thus, formula (2.1.13) gives us the solution of the problem (2.1.12). For $x \geq 0, t \geq 0$ we rewrite (2.1.3) in the form

$$u(x,t) = \begin{cases} \frac{1}{2}\left(\varphi(x+at) + \varphi(x-at)\right) + \frac{1}{2a}\int_{x-at}^{x+at} \psi(s)\, ds, & x \geq at, \\ \frac{1}{2}\left(\varphi(x+at) - \varphi(at-x)\right) + \frac{1}{2a}\int_{at-x}^{x+at} \psi(s)\, ds, & x \leq at. \end{cases} \tag{2.1.14}$$

We note that in the domain $x \geq at$, the influence of the wave reflected from the end $x = 0$ is not present, and the solution has the same form as for the infinite string.

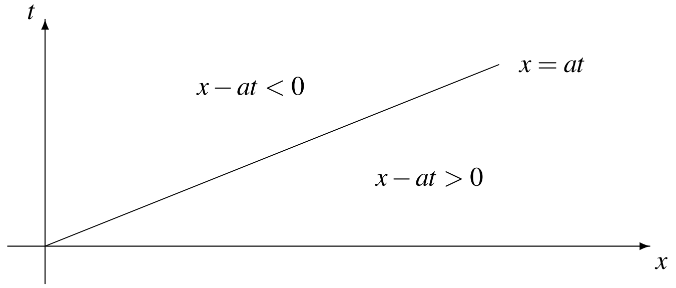

Figure 2.1.3.

Remark 2.1.3. Analogously one can solve the problem with the boundary condition $u_{x|x=0} = 0$. This can be recommended as an exercise. In this case, according to Lemma 2.1.1 we should extend $\varphi(x)$ and $\psi(x)$ as even functions. Other exercises related to the topic one can find in Chapter 6, section 6.2.

2.2. The Mixed Problem for the Equation of the Vibrating String

This section deals with mixed problems for the equation of the vibrating string on a finite interval. For studying such problems we use the *method of separation of variables,* which is also called the *method of standing waves.* Subsections 1-3 are related to the standard level for a course of partial differential equations of mathematical physics. Subsection 4 describes a general scheme of separation of variables, and it contains more complicated material. This subsection can be recommended for advanced studies and can be omitted "in the first reading".

1. Homogeneous equation

We consider the following mixed problem

$$u_{tt} = a^2 u_{xx}, \quad 0 < x < l,\ t > 0, \tag{2.2.1}$$

$$u_{|x=0} = u_{|x=l} = 0, \tag{2.2.2}$$

$$u_{|t=0} = \varphi(x), \quad u_{t|t=0} = \psi(x). \tag{2.2.3}$$

This problem describes the oscillation process for a finite string of the length l, fixed at the ends $x = 0$ and $x = l$, provided that the initial displacement $\varphi(x)$ and the initial speed $\psi(x)$ are known. Other mixed problems (including physical motivations) can be found in Section 6.2.

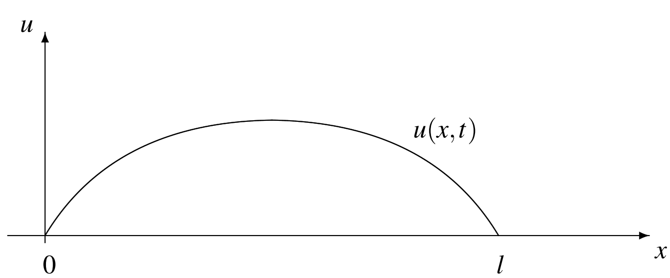

Figure 2.2.1.

Denote $D = \{(x,t) :\ 0 \le x \le l, t \ge 0\}$.

Definition 2.2.1. A function $u(x,t)$ is called a solution of problem (2.2.1)-(2.2.3) if $u(x,t) \in C^2(D)$, and $u(x,t)$ satisfies (2.2.1)-(2.2.3).

First we consider the following auxiliary problem. We will seek nontrivial (i.e. not identically zero) particular solutions of equation (2.2.1) such that they satisfy the boundary conditions (2.2.2) and admit separation of variables, i.e. they have the form

$$u(x,t) = Y(x)T(t). \tag{2.2.4}$$

For this purpose we substitute (2.2.4) into (2.2.1) and (2.2.2). By virtue of (2.2.1) we have

$$Y(x)\ddot{T}(t) = a^2 Y''(x)T(t).$$

Here and below, " $'$ " denotes differentiation with respect to x, and " $\cdot$ " denotes differentiation with respect to t. Separating the variables, we obtain

$$\frac{Y''(x)}{Y(x)} = \frac{\ddot{T}(t)}{a^2 T(t)}.$$

Since x and t are independent variables, the equality is here valid only if both fractions are equal to a constant; we denote this constant by $-\lambda$. Therefore,

$$\frac{Y''(x)}{Y(x)} = \frac{\ddot{T}(t)}{a^2 T(t)} = -\lambda. \tag{2.2.5}$$

Moreover, the boundary conditions (2.2.2) yield

$$\left.\begin{array}{l} 0 = u(0,t) = Y(0)T(t), \\ 0 = u(l,t) = Y(l)T(t). \end{array}\right\} \tag{2.2.6}$$

It follows from (2.2.5) and (2.2.6) that

$$\ddot{T}(t) + a^2\lambda T(t) = 0, \tag{2.2.7}$$

$$Y''(x) + \lambda Y(x) = 0, \tag{2.2.8}$$

$$Y(0) = Y(l) = 0. \tag{2.2.9}$$

Thus, the function $T(t)$ is a solution of the ordinary differential equation (2.2.7), and the function $Y(x)$ is a solution of the boundary value problem (2.2.8)-(2.2.9) which is called the associated *Sturm-Liouville problem*. We are interested in nontrivial solutions of problem (2.2.8)-(2.2.9), but it turns out that they exist only for some particular values of the parameter λ.

Definition 2.2.2. The values of the parameter λ for which the problem (2.2.8)-(2.2.9) has nonzero solutions are called *eigenvalues*, and the corresponding nontrivial solutions are called *eigenfunctions*. The set of eigenvalues is called *the spectrum* of the Sturm-Liouville problem.

Clearly, the eigenfunctions are defined up to a multiplicative constant, since if $Y(x)$ is a nontrivial solution of the problem (2.2.8)-(2.2.9) for $\lambda = \lambda^0$ (i.e. $Y(x)$ is an eigenfunction corresponding to the eigenvalue λ^0), then the function $CY(x)$, $C-const$, is also a solution of problem (2.2.8)-(2.2.9) for the same value $\lambda = \lambda^0$.

Let us find the eigenvalues and eigenfunctions of problem (2.2.8)-(2.2.9). Let $\lambda = \rho^2$. The general solution of equation (2.2.8) for each fixed λ has the form

$$Y(x) = A\frac{\sin\rho x}{\rho} + B\cos\rho x \tag{2.2.10}$$

(in particular, for $\rho = 0$ (2.2.10) yields $Y(x) = Ax + B$). Since $Y(0) = 0$, in (2.2.10) $B = 0$, i.e.

$$Y(x) = A\frac{\sin\rho x}{\rho}. \tag{2.2.11}$$

Substituting (2.2.11) into the boundary condition $Y(l) = 0$, we obtain

$$A\frac{\sin\rho l}{\rho} = 0.$$

Since we seek nontrivial solutions, $A \neq 0$, and we arrive at the following equation for the eigenvalues:

$$\frac{\sin\rho l}{\rho} = 0. \tag{2.2.12}$$

The roots of equation (2.2.12) are $\rho_n = \dfrac{n\pi}{l}$, and consequently, the eigenvalues of problem (2.2.8)-(2.2.9) have the form

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad n = 1, 2, 3, \ldots, \tag{2.2.13}$$

and the corresponding eigenfunctions, by virtue of (2.2.11), are

$$Y_n(x) = \sin\frac{n\pi}{l}x, \quad n = 1, 2, 3, \ldots \tag{2.2.14}$$

(up to a multiplicative constant). Since nontrivial solutions of problem (2.2.8)-(2.2.9) exist only for $\lambda = \lambda_n$ of the form (2.2.13), it makes sense to consider equation (2.2.7) only for $\lambda = \lambda_n$:

$$\ddot{T}_n(t) + a^2\lambda_n T_n(t) = 0.$$

The general solution of this equation has the form

$$T_n(t) = A_n \sin\frac{an\pi}{l}t + B_n\cos\frac{an\pi}{l}t, \quad n \geq 1, \tag{2.2.15}$$

where A_n and B_n are arbitrary constants. Substituting (2.2.14) and (2.2.15) into (2.2.4) we obtain all solutions of the auxiliary problem:

$$u_n(x,t) = \left(A_n \sin\frac{an\pi}{l}t + B_n\cos\frac{an\pi}{l}t\right)\sin\frac{n\pi}{l}x, \quad n = 1, 2, 3, \ldots \tag{2.2.16}$$

The solutions of the form (2.2.16) are called *standing waves* (normal modes of vibrations).

We will seek the solution of the mixed problem (2.2.1)-(2.2.3) by superposition of standing waves (2.2.16), i.e. as a *formal series solution*:

$$u(x,t) = \sum_{n=1}^{\infty}\left(A_n \sin\frac{an\pi}{l}t + B_n\cos\frac{an\pi}{l}t\right)\sin\frac{n\pi}{l}x. \tag{2.2.17}$$

By construction the function $u(x,t)$ satisfies formally equation (2.2.1) and the boundary conditions (2.2.2) for all A_n and B_n. Choose A_n and B_n such that $u(x,t)$ satisfies also the initial conditions (2.2.3). Substituting (2.2.17) into (2.2.3) we calculate

$$\varphi(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{l}x,$$

$$\psi(x) = \sum_{n=1}^{\infty} \frac{an\pi}{l} A_n \sin \frac{n\pi}{l} x.$$

Using the formulae for the Fourier coefficients [4, Chapter 1], we obtain

$$\left. \begin{aligned} B_n &= \frac{2}{l} \int_0^l \varphi(x) \sin \frac{n\pi}{l} x \, dx, \\ A_n &= \frac{2}{an\pi} \int_0^l \psi(x) \sin \frac{n\pi}{l} x \, dx. \end{aligned} \right\} \tag{2.2.18}$$

Let us now show that the function $u(x,t)$, defined by (2.2.17)-(2.2.18), is the solution of the mixed problem (2.2.1)-(2.2.3). For this purpose we need some conditions on $\varphi(x)$ and $\psi(x)$.

Theorem 2.2.1. *Let* $\varphi(x) \in C^3[0,l]$, $\psi(x) \in C^2[0,l]$, $\varphi(0) = \varphi(l) = \varphi''(0) = \varphi''(l) = \psi(0) = \psi(l) = 0$. *Then the function* $u(x,t)$, *defined by* $(2.2.17)-(2.2.18)$, *is the solution of the mixed problem* $(2.2.1)-(2.2.3)$.

Proof. From the theory of the Fourier series [11, Chapter 1] we know the following facts:

1) The system of functions

$$\left\{ \sin \frac{n\pi}{l} x \right\}_{n \geq 1}$$

is complete and orthogonal in $L_2(0,l)$.
2) Let $f(x) \in L_2(0,l)$, and let

$$\alpha_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi}{l} x \, dx,$$

$$\beta_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi}{l} x \, dx$$

be the Fourier coefficients for the function $f(x)$. Then

$$\sum_n \left(|\alpha_n|^2 + |\beta_n|^2 \right) < \infty. \tag{2.2.19}$$

Moreover, we note:

$$\text{If numbers } \gamma_n \text{ are such that } \sum_n |\gamma_n|^2 < \infty, \text{ then } \sum_n \frac{|\gamma_n|}{n} < \infty. \tag{2.2.20}$$

This follows from the obvious inequality

$$\frac{1}{n} |\gamma_n| \leq \frac{1}{2} \left(\frac{1}{n^2} + |\gamma_n^2| \right).$$

Let us show that

$$\sum_{n=1}^{\infty} n^2 (|A_n| + |B_n|) < \infty. \tag{2.2.21}$$

Indeed, integrating by parts the integrals in (2.2.18), we get

$$B_n = \frac{2}{l}\int_0^l \varphi(x)\sin\frac{n\pi}{l}x\,dx = \frac{2}{\pi n}\int_0^l \varphi'(x)\cos\frac{n\pi}{l}x\,dx$$

$$= -\frac{2l}{\pi^2 n^2}\int_0^l \varphi''(x)\sin\frac{n\pi}{l}x\,dx = -\frac{2l^2}{\pi^3 n^3}\int_0^l \varphi'''(x)\cos\frac{n\pi}{l}x\,dx.$$

Similarly, we calculate

$$A_n = -\frac{2l^2}{a\pi^3 n^3}\int_0^l \psi''(x)\sin\frac{n\pi}{l}x\,dx.$$

Hence, using proposition (2.2.20) and the properties of the Fourier coefficients (2.2.19), we arrive at (2.2.21). It follows from (2.2.21) that the function $u(x,t)$, defined by (2.2.17)-(2.2.18), has in D continuous partial derivatives up to the second order (i.e. $u(x,t) \in C^2(D)$), and these derivatives can be calculated by termwise differentiation of the series (2.2.17). Obviously, $u(x,t)$ is a solution of equation (2.2.1) and satisfies the boundary conditions (2.2.2). Let us show that $u(x,t)$ satisfies also the initial conditions (2.2.3). Indeed, it follows from (2.2.17) for $t=0$ that

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{l}x,$$

$$u_t(x,0) = \sum_{n=1}^{\infty} \frac{a\pi n}{l} A_n \sin\frac{n\pi}{l}x,$$

and consequently,

$$B_n = \frac{2}{l}\int_0^l u(x,0)\sin\frac{n\pi}{l}x\,dx,$$

$$A_n = \frac{2}{an\pi}\int_0^l u_t(x,0)\sin\frac{n\pi}{l}x\,dx.$$

Comparing these relations with (2.2.18) we obtain for all $n \ge 1$:

$$\int_0^l (u(x,0)-\varphi(x))\sin\frac{n\pi}{l}x\,dx = 0,$$

$$\int_0^l (u_t(x,0)-\psi(x))\sin\frac{n\pi}{l}x\,dx = 0.$$

By virtue of the completeness of the system of functions $\left\{\sin\frac{n\pi}{l}x\right\}_{n\ge 1}$ in $L_2(0,l)$, we conclude that

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x),$$

i.e. the function $u(x,t)$ satisfies the initial conditions (2.2.3). □

2. Uniqueness of the solution of the mixed problem and the energy integral

Theorem 2.2.2. *If the solution of the mixed problem* $(2.2.1)-(2.2.3)$ *exists, then it is unique.*

Proof. Let $u_1(x,t)$ and $u_2(x,t)$ be solutions of (2.2.1)-(2.2.3). Denote

$$u(x,t) := u_1(x,t) - u_2(x,t).$$

Then $u(x,t) \in C^2(D)$ and

$$\left.\begin{array}{c} u_{tt} = a^2 u_{xx}, \\ u_{|x=0} = u_{|x=l} = 0, \quad u_{|t=0} = u_{t|t=0} = 0. \end{array}\right\} \tag{2.2.22}$$

We consider the integral

$$E(t) = \frac{1}{2}\int_0^l \left(\frac{1}{a^2}(u_t(x,t))^2 + (u_x(x,t))^2\right) dx,$$

which is called the *energy integral*. We calculate

$$\dot{E}(t) = \int_0^l \left(\frac{1}{a^2}u_t(x,t)u_{tt}(x,t) + u_x(x,t)u_{xt}(x,t)\right) dx,$$

where $\dot{E}(t) := \frac{d}{dt}E(t)$. Integrating by parts the second term in the last integral we obtain

$$\dot{E}(t) = \int_0^l u_t(x,t)\left(\frac{1}{a^2}u_{tt}(x,t) - u_{xx}(x,t)\right) dx + \Big|_0^l \left(u_x(x,t)u_t(x,t)\right).$$

By virtue of (2.2.22),

$$\frac{1}{a^2}u_{tt}(x,t) - u_{xx}(x,t) \equiv 0,$$

and consequently,

$$\dot{E}(t) = (u_x u_t)_{|x=l} - (u_x u_t)_{|x=0}.$$

According to (2.2.22) $u(0,t) \equiv 0$ and $u(l,t) \equiv 0$. Hence $u_{t|x=0} = 0$ and $u_{t|x=l} = 0$. This yields $\dot{E}(t) \equiv 0$, and consequently, $E(t) \equiv const$. Since

$$E(0) = \frac{1}{2}\int_0^l \left(\frac{1}{a^2}(u_t(x,0)^2 + (u_x(x,0)^2\right) dx = 0,$$

we get $E(t) \equiv 0$. This implies

$$u_t(x,t) \equiv 0, \quad u_x(x,t) \equiv 0,$$

i.e. $u(x,t) \equiv const$. Using (2.2.22) again we obtain $u(x,t) \equiv 0$, and Theorem 2.2.2 is proved. □

3. Non-homogeneous equation

We consider the mixed problem for the *non-homogeneous* equation of the vibrating string:

$$\left.\begin{array}{c} u_{tt} = a^2 u_{xx} + f(x,t), \quad 0 < x < l,\ t > 0, \\ u_{|x=0} = u_{|x=l} = 0, \\ u_{|t=0} = \varphi(x), \quad u_{t|t=0} = \psi(x). \end{array}\right\} \tag{2.2.23}$$

Let the function $f(x,t)$ be continuous in $D,$ twice continuously differentiable with respect to $x,$ and $f(0,t) = f(l,t) = 0.$ Let the functions $\varphi(x)$ and $\psi(x)$ satisfy the conditions of Theorem 2.2.1. We expand the function $f(x,t)$ into a Fourier series with respect to x:

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin \frac{n\pi}{l} x,$$

where

$$f_n(t) = \frac{2}{l} \int_0^l f(x,t) \sin \frac{n\pi}{l} x\, dx.$$

We will seek the solution of the problem (2.2.23) as a series in eigenfunctions of the Sturm-Liouville problem (2.2.8)-(2.2.9):

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin \frac{n\pi}{l} x. \tag{2.2.24}$$

Substituting (2.2.24) into (2.2.23) we get

$$\sum_{n=1}^{\infty} \ddot{u}_n(t) \sin \frac{n\pi}{l} x = -a^2 \sum_{n=1}^{\infty} \left(\frac{\pi n}{l}\right)^2 u_n(t) \sin \frac{n\pi}{l} x + \sum_{n=1}^{\infty} f_n(t) \sin \frac{n\pi}{l} x,$$

$$\sum_{n=1}^{\infty} u_n(0) \sin \frac{n\pi}{l} x = \varphi(x), \quad \sum_{n=1}^{\infty} \dot{u}_n(0) \sin \frac{n\pi}{l} x = \psi(x),$$

and consequently, we obtain the following relations for finding the functions $u_n(t)$, $n \ge 1$:

$$\left.\begin{array}{c} \ddot{u}_n(t) + \left(\dfrac{a\pi n}{l}\right)^2 u_n(t) = f_n(t), \\ u_n(0) = B_n,\ \dot{u}_n(0) = \dfrac{a\pi n}{l} A_n, \end{array}\right\} \tag{2.2.25}$$

where the numbers A_n and B_n are defined by (2.2.18). The Cauchy problem (2.2.25) has a unique solution which can be found, for example, by the method of variations of constants:

$$u_n(t) = A_n \sin \frac{an\pi}{l} t + B_n \cos \frac{an\pi}{l} t$$

$$+ \frac{l}{an\pi} \int_0^t f_n(\tau) \sin \frac{an\pi}{l} (t - \tau)\, d\tau. \tag{2.2.26}$$

It is easy to check that the function $u(x,t),$ defined by (2.2.24) and (2.2.26), is a solution of problem (2.2.23).

4. A general scheme of separation of variables

This subsection is recommended for advanced studies, and it can be omitted ”in the first reading”.

4.1. The method of separation of variables can be used for a wide class of differential equations of mathematical physics. In this section we apply the method of separation of variables for studying the equation of a non-homogeneous vibrating string. We consider the following mixed problem

$$\rho(x)u_{tt} = \Big(k(x)u_x\Big)_x - q(x)u, \quad 0 < x < l,\ t > 0, \tag{2.2.27}$$

$$(h_1 u_x - hu)_{|x=0} = 0, \quad (H_1 u_x + Hu)_{|x=l} = 0, \tag{2.2.28}$$

$$u_{|t=0} = \Phi(x), \quad u_{t|t=0} = \Psi(x). \tag{2.2.29}$$

Here x and t are independent variables, $u(x,t)$ is an unknown function, the functions ρ, k, q, Φ, Ψ are real-valued, and the numbers h, h_1, H, H_1 are real. We assume that $q(x) \in L_2(0,l)$, $\rho(x), k(x), \Phi(x), \Psi(x) \in W_2^2[0,l]$, $\rho(x) > 0, k(x) > 0$, and Φ, Ψ satisfy the boundary conditions (2.2.28). We remind that the condition $f(x) \in W_2^2[0,l]$ means that $f(x)$ and $f'(x)$ are absolutely continuous on $[0,l]$, and $f''(x) \in L_2(0,l)$.

First we consider the following auxiliary problem. We will seek nontrivial (i.e. not identically zero) particular solutions of equation (2.2.27) such that they satisfy the boundary conditions (2.2.28) and admit separation of variables, i.e. they have the form

$$u(x,t) = Y(x)T(t). \tag{2.2.30}$$

For this purpose we substitute (2.2.30) into (2.2.27) and (2.2.28). By virtue of (2.2.27) we have

$$\rho(x)Y(x)\ddot{T}(t) = \Big(\Big(k(x)Y'(x)\Big)' - q(x)Y(x)\Big)T(t).$$

Separating the variables, we obtain

$$\frac{\Big(k(x)Y'(x)\Big)' - q(x)Y(x)}{\rho(x)Y(x)} = \frac{\ddot{T}(t)}{T(t)}.$$

Since x and t are independent variables, the equality is here valid only if both fractions are equal to a constant; we denote this constant by $-\lambda$. Therefore,

$$\frac{\Big(k(x)Y'(x)\Big)' - q(x)Y(x)}{\rho(x)Y(x)} = \frac{\ddot{T}(t)}{T(t)} = -\lambda. \tag{2.2.31}$$

Moreover, the boundary conditions (2.2.28) yield

$$\left.\begin{aligned} \Big(h_1 Y'(0) - hY(0)\Big)T(t) = 0, \\ \Big(H_1 Y'(l) + HY(l)\Big)T(t) = 0. \end{aligned}\right\} \tag{2.2.32}$$

It follows from (2.2.31) and (2.2.32) that

$$\ddot{T}(t)+\lambda T(t)=0, \tag{2.2.33}$$

$$-\Big(k(x)Y'(x)\Big)'+q(x)Y(x)=\lambda\rho(x)Y(x), \quad 0<x<l, \tag{2.2.34}$$

$$h_1Y'(0)-hY(0)=0, \quad H_1Y'(l)+HY(l)=0. \tag{2.2.35}$$

Thus, the function $T(t)$ is a solution of the ordinary differential equation (2.2.33), and the function $Y(x)$ is a solution of the boundary value problem (2.2.34)-(2.2.35) which is called the *Sturm-Liouville problem* associated to the mixed problem (2.2.27)-(2.2.29). We are interested in nontrivial solutions of problem (2.2.34)-(2.2.35), but it turns out that they exist only for some particular values of the parameter λ.

Definition 2.2.3. The values of the parameter λ for which the problem (2.2.34)-(2.2.35) has nonzero solutions are called *eigenvalues*, and the corresponding nontrivial solutions are called *eigenfunctions*. The set of eigenvalues is called *the spectrum* of the Sturm-Liouville problem.

medskip We note that Definition 2.2.3 is a generalization of Definition 2.2.2, since it is related to the more general problem (2.2.34)-(2.2.35) than problem (2.2.8)-(2.2.9).

Clearly, the eigenfunctions are defined up to a multiplicative constant, since if $Y(x)$ is a nontrivial solution of the problem (2.2.34)-(2.2.35) for $\lambda=\lambda^0$ (i.e. $Y(x)$ is an eigenfunction corresponding to the eigenvalue λ^0), then the function $CY(x)$, $C-const$, is also a solution of problem (2.2.34)-(2.2.35) for the same value $\lambda=\lambda^0$.

4.2. In this subsection we establish properties of the eigenvalues and the eigenfunctions of the boundary value problem (2.2.34)-(2.2.35). For definiteness, let below $h_1H_1\neq 0$. The other cases are considered analogously and can be recommended as exercises.

First we study the particular case when

$$\rho(x)=k(x)\equiv 1, \quad l=\pi, \tag{2.2.36}$$

and then we will show that the general case can be reduced to (2.2.36). In other words, one can consider (2.2.36) without loss of generality. Thus, we consider the Sturm-Liouville problem L in the following form:

$$-y''(x)+q(x)y(x)=\lambda y(x), \quad 0<x<\pi, \tag{2.2.37}$$

$$y'(0)-hy(0)=0, \quad y'(\pi)+Hy(\pi)=0, \tag{2.2.38}$$

where $q(x)\in L_2(0,\pi)$, h and H are real. Denote

$$\ell y(x):=-y''(x)+q(x)y(x),$$

$$U(y):=y'(0)-hy(0),$$

$$V(y):=y'(\pi)+Hy(\pi).$$

Let $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ be solutions of (2.2.37) under the initial conditions

$$\varphi(0,\lambda)=1, \quad \varphi'(0,\lambda)=h, \quad \psi(\pi,\lambda)=1, \quad \psi'(\pi,\lambda)=-H.$$

For each fixed x, the functions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ are entire in λ. Clearly,

$$\left.\begin{aligned} U(\varphi) = \varphi'(0,\lambda) - h\varphi(0,\lambda) = 0, \\ V(\psi) = \psi'(\pi,\lambda) + H\psi(\pi,\lambda) = 0. \end{aligned}\right\} \tag{2.2.39}$$

Denote

$$\Delta(\lambda) = \langle \psi(x,\lambda), \varphi(x,\lambda) \rangle, \tag{2.2.40}$$

where

$$\langle y(x), z(x) \rangle := y(x)z'(x) - y'(x)z(x)$$

is the Wronskian of y and z. By virtue of Liouville‘s formula for the Wronskian, $\langle \psi(x,\lambda), \varphi(x,\lambda) \rangle$ does not depend on x. The function $\Delta(\lambda)$ is called *the characteristic function* of L. Substituting $x = 0$ and $x = \pi$ into (2.2.40), we get

$$\Delta(\lambda) = V(\varphi) = -U(\psi). \tag{2.2.41}$$

The function $\Delta(\lambda)$ is entire in λ, and consequently, it has an at most countable set of zeros $\{\lambda_n\}$ (see [12]).

Theorem 2.2.3. *The zeros $\{\lambda_n\}$ of the characteristic function coincide with the eigenvalues of the boundary value problem L. The functions $\varphi(x,\lambda_n)$ and $\psi(x,\lambda_n)$ are eigenfunctions, and there exists a sequence $\{\beta_n\}$ such that*

$$\psi(x,\lambda_n) = \beta_n \varphi(x,\lambda_n), \quad \beta_n \neq 0. \tag{2.2.42}$$

Proof. 1) Let λ_0 be a zero of $\Delta(\lambda)$. Then, by virtue of (2.2.39)-(2.2.41), $\psi(x,\lambda_0) = \beta_0 \varphi(x,\lambda_0)$, and the functions $\psi(x,\lambda_0), \varphi(x,\lambda_0)$ satisfy the boundary conditions (2.2.38). Hence, λ_0 is an eigenvalue, and $\psi(x,\lambda_0), \varphi(x,\lambda_0)$ are eigenfunctions related to λ_0.

2) Let λ_0 be an eigenvalue of L, and let y_0 be a corresponding eigenfunction. Then $U(y_0) = V(y_0) = 0$. Clearly $y_0(0) \neq 0$ (if $y_0(0) = 0$ then $y_0'(0) = 0$ and, by virtue of the uniqueness theorem for the equation (2.2.37), $y_0(x) \equiv 0$). Without loss of generality we put $y_0(0) = 1$. Then $y_0'(0) = h$, and consequently $y_0(x) \equiv \varphi(x,\lambda_0)$. Therefore, (2.2.41) yields

$$\Delta(\lambda_0) = V(\varphi(x,\lambda_0)) = V(y_0(x)) = 0,$$

and Theorem 2.2.3 is proved. □

Note that we have also proved that for each eigenvalue there exists only one (up to a multiplicative constant) eigenfunction.

Denote

$$\alpha_n := \int_0^\pi \varphi^2(x,\lambda_n)\,dx. \tag{2.2.43}$$

The numbers $\{\alpha_n\}$ are called *the weight numbers,* and the numbers $\{\lambda_n, \alpha_n\}$ are called *the spectral data* of L.

Lemma 2.2.1. *The following relation holds*

$$\beta_n \alpha_n = -\dot{\Delta}(\lambda_n), \tag{2.2.44}$$

where the numbers β_n *are defined by* (2.2.42), *and* $\dot{\Delta}(\lambda) = \dfrac{d}{d\lambda}\Delta(\lambda)$.

Proof. Since

$$-\psi''(x,\lambda) + q(x)\psi(x,\lambda) = \lambda\psi(x,\lambda),$$
$$-\varphi''(x,\lambda_n) + q(x)\varphi(x,\lambda_n) = \lambda_n\varphi(x,\lambda_n),$$

we get

$$\psi(x,\lambda)\varphi''(x,\lambda_n) - \psi''(x,\lambda)\varphi(x,\lambda_n) = (\lambda-\lambda_n)\psi(x,\lambda)\varphi(x,\lambda_n),$$

and consequently,

$$\frac{d}{dx}\langle\psi(x,\lambda),\varphi(x,\lambda_n)\rangle = (\lambda-\lambda_n)\psi(x,\lambda)\varphi(x,\lambda_n).$$

After integration we obtain

$$(\lambda-\lambda_n)\int_0^\pi \psi(x,\lambda)\varphi(x,\lambda_n)\,dx = \langle\psi(x,\lambda),\varphi(x,\lambda_n)\rangle\Big|_0^\pi,$$

and hence with the help of (2.2.41),

$$(\lambda-\lambda_n)\int_0^\pi \psi(x,\lambda)\varphi(x,\lambda_n)\,dx$$
$$= \varphi'(\pi,\lambda_n) + H\varphi(\pi,\lambda_n) + \psi'(0,\lambda) - h\psi(0,\lambda) = -\Delta(\lambda).$$

For $\lambda \to \lambda_n$, this yields

$$\int_0^\pi \psi(x,\lambda_n)\varphi(x,\lambda_n)\,dx = -\dot{\Delta}(\lambda_n).$$

Using (2.2.42) and (2.2.43) we arrive at (2.2.44). □

Theorem 2.2.4. *The eigenvalues* λ_n *and the eigenfunctions* $\varphi(x,\lambda_n)$ *and* $\psi(x,\lambda_n)$ *of* L *are real. All zeros of* $\Delta(\lambda)$ *are simple, i.e.* $\dot{\Delta}(\lambda_n) \neq 0$. *Eigenfunctions related to different eigenvalues are orthogonal in* $L_2(0,\pi)$.

Proof. Let λ_n and λ_k ($\lambda_n \neq \lambda_k$) be eigenvalues with eigenfunctions $y_n(x)$ and $y_k(x)$ respectively. Then integration by parts yields

$$\int_0^\pi \ell y_n(x)\,y_k(x)\,dx = -\int_0^\pi y_n''(x)y_k(x)\,dx + \int_0^\pi q(x)y_n(x)y_k(x)\,dx$$
$$= \Big|_0^\pi \big(y_n(x)y_k'(x) - y_n'(x)y_k(x)\big)$$
$$-\int_0^\pi y_n(x)y_k''(x)\,dx + \int_0^\pi q(x)y_n(x)y_k(x)\,dx.$$

The substitution vanishes since the eigenfunctions $y_n(x)$ and $y_k(x)$ satisfy the boundary conditions (2.2.38). Therefore,

$$\int_0^\pi \ell y_n(x)\,y_k(x)\,dx = \int_0^\pi y_n(x)\ell y_k(x)\,dx,$$

and hence

$$\lambda_n \int_0^\pi y_n(x)y_k(x)\,dx = \lambda_k \int_0^\pi y_n(x)y_k(x)\,dx,$$

or

$$\int_0^\pi y_n(x)y_k(x)\,dx = 0.$$

This proves that eigenfunctions related to distinct eigenvalues are orthogonal in $L_2(0,\pi)$.

Further, let $\lambda^0 = u+iv, v \neq 0$ be a non-real eigenvalue with an eigenfunction $y^0(x) \neq 0$. Since $q(x), h$ and H are real, we get that $\overline{\lambda^0} = u - iv$ is also the eigenvalue with the eigenfunction $\overline{y^0(x)}$. Since $\lambda^0 \neq \overline{\lambda^0}$, we derive as before

$$\|y^0\|_{L_2}^2 = \int_0^\pi y^0(x)\overline{y^0(x)}\,dx = 0,$$

which is impossible. Thus, all eigenvalues $\{\lambda_n\}$ of L are real, and consequently the eigenfunctions $\varphi(x,\lambda_n)$ and $\psi(x,\lambda_n)$ are real too. Since $\alpha_n \neq 0$, $\beta_n \neq 0$, we get by virtue of (2.2.44) that $\dot{\Delta}(\lambda_n) \neq 0$. □

Example. Let $q(x) = 0$, $h = 0$ and $H = 0$, and let $\lambda = \rho^2$. Then one can check easily that

$$\varphi(x,\lambda) = \cos\rho x, \quad \psi(x,\lambda) = \cos\rho(\pi - x),$$

$$\Delta(\lambda) = -\rho\sin\rho\pi,\ \lambda_n = n^2\ (n \geq 0),\ \varphi(x,\lambda_n) = \cos nx,\ \beta_n = (-1)^n.$$

Lemma 2.2.2. *For* $|\rho| \to \infty$, *the following asymptotic formulae hold*

$$\left.\begin{aligned} \varphi(x,\lambda) &= \cos\rho x + O\left(\frac{1}{|\rho|}\exp(|\tau|x)\right), \\ \varphi'(x,\lambda) &= -\rho\sin\rho x + O(\exp(|\tau|x)), \end{aligned}\right\} \tag{2.2.45}$$

$$\left.\begin{aligned} \psi(x,\lambda) &= \cos\rho(\pi - x) + O\left(\frac{1}{|\rho|}\exp(|\tau|(\pi - x))\right), \\ \psi'(x,\lambda) &= \rho\sin\rho(\pi - x) + O(\exp(|\tau|(\pi - x))), \end{aligned}\right\} \tag{2.2.46}$$

uniformly with respect to $x \in [0,\pi]$. Here and in the sequel, $\lambda = \rho^2$, $\tau = Im\,\rho$, and o and O denote the Landau symbols (see [13], §4).

Proof. Let us show that

$$\varphi(x,\lambda) = \cos\rho x + h\frac{\sin\rho x}{\rho} + \int_0^x \frac{\sin\rho(x-t)}{\rho} q(t)\varphi(t,\lambda)\,dt. \tag{2.2.47}$$

Indeed, the Volterra integral equation

$$y(x,\lambda) = \cos\rho x + h\frac{\sin\rho x}{\rho} + \int_0^x \frac{\sin\rho(x-t)}{\rho} q(t)y(t,\lambda)\,dt$$

has a unique solution (see [14]). On the other hand, if a certain function $y(x,\lambda)$ satisfies this equation, then we get by differentiation

$$y'(x,\lambda) = -\rho\sin\rho x + h\cos\rho x + \int_0^x \cos\rho(x-t) q(t)y(t,\lambda)\,dt,$$

$$y''(x,\lambda) = -\rho^2\cos\rho x - h\rho\sin\rho x + q(x)y(x,\lambda)$$
$$-\int_0^x \rho\sin\rho(x-t)q(t)y(t,\lambda)\,dt,$$

and consequently,

$$y''(x,\lambda) + \rho^2 y(x,\lambda) = q(x)y(x,\lambda),$$
$$y(0,\lambda) = 1, \quad y'(0,\lambda) = h,$$

i.e. $y(x,\lambda) \equiv \varphi(x,\lambda)$, and (2.2.47) is valid.

Differentiating (2.2.47) we calculate

$$\varphi'(x,\lambda) = -\rho\sin\rho x + h\cos\rho x + \int_0^x \cos\rho(x-t)q(t)\varphi(t,\lambda)\,dt. \tag{2.2.48}$$

Denote

$$\mu(\lambda) = \max_{0\le x\le\pi}(|\varphi(x,\lambda)|\exp(-|\tau|x)).$$

Since $|\sin\rho x| \le \exp(|\tau|x)$ and $|\cos\rho x| \le \exp(|\tau|x)$, we have from (2.2.47) that for $|\rho| \ge 1,\ x \in [0,\pi]$,

$$|\varphi(x,\lambda)|\exp(-|\tau|x) \le 1 + \frac{1}{|\rho|}\Big(h + \mu(\lambda)\int_0^x |q(t)|\,dt\Big) \le C_1 + \frac{C_2}{|\rho|}\mu(\lambda),$$

and consequently

$$\mu(\lambda) \le C_1 + \frac{C_2}{|\rho|}\mu(\lambda)$$

or

$$\mu(\lambda) \le C_1\left(1 + \frac{C_2}{|\rho|}\right)^{-1}.$$

For sufficiently large $|\rho|$, this yields $\mu(\lambda) = O(1)$, i.e. $\varphi(x,\lambda) = O(\exp(|\tau|x))$. Substituting this estimate into the right-hand sides of (2.2.47) and (2.2.48), we arrive at (2.2.45). Similarly one can derive (2.2.46).

We note that (2.2.46) can be also obtained directly from (2.2.45). Indeed, since

$$-\psi''(x,\lambda) + q(x)\psi(x,\lambda) = \lambda\psi(x,\lambda),$$
$$\psi(\pi,\lambda) = 1, \quad \psi'(\pi,\lambda) = -H,$$

the function $\tilde\varphi(x,\lambda) := \psi(\pi - x,\lambda)$ satisfies the following differential equation and the initial conditions

$$-\tilde\varphi''(x,\lambda) + q(\pi - x)\tilde\varphi(x,\lambda) = \lambda\tilde\varphi(x,\lambda),$$
$$\tilde\varphi(0,\lambda) = 1, \quad \tilde\varphi'(0,\lambda) = H.$$

Therefore, the asymptotic formulae (2.2.45) are also valid for the function $\tilde\varphi(x,\lambda)$. From this we arrive at (2.2.46). □

Theorem 2.2.5. *The boundary value problem* L *has a countable set of eigenvalues* $\{\lambda_n\}_{n\ge0}$. *For* $n \to \infty$,

$$\rho_n = \sqrt{\lambda_n} = n + \frac{\omega}{\pi n} + \frac{\kappa_n}{n}, \quad \{\kappa_n\} \in l_2, \tag{2.2.49}$$

$$\varphi(x,\lambda_n) = \cos nx + \frac{\xi_n(x)}{n}, \quad |\xi_n(x)| \le C, \tag{2.2.50}$$

where

$$\omega = h + H + \frac{1}{2}\int_0^\pi q(t)\,dt.$$

Here and everywhere below one and the same symbol $\{\kappa_n\}$ denotes various sequences from l_2, and the symbol C denotes various positive constants which do not depend on x, λ and n. We remind that $\{\kappa_n\} \in l_2$ means $\sum_n |\kappa_n|^2 < \infty$ (see e.g. [13]).

Proof. 1) Substituting the asymptotics for $\varphi(x,\lambda)$ from (2.2.45) into the right-hand sides of (2.2.47) and (2.2.48), we calculate

$$\varphi(x,\lambda) = \cos\rho x + q_1(x)\frac{\sin\rho x}{\rho} + \frac{1}{2\rho}\int_0^x q(t)\sin\rho(x-2t)\,dt + O\Big(\frac{\exp(|\tau|x)}{\rho^2}\Big),$$

$$\varphi'(x,\lambda) = -\rho\sin\rho x + q_1(x)\cos\rho x + \frac{1}{2}\int_0^x q(t)\cos\rho(x-2t)\,dt + O\Big(\frac{\exp(|\tau|x)}{\rho}\Big),$$

where

$$q_1(x) = h + \frac{1}{2}\int_0^x q(t)\,dt.$$

According to (2.2.41), $\Delta(\lambda) = \varphi'(\pi,\lambda) + H\varphi(\pi,\lambda)$. Hence,

$$\Delta(\lambda) = -\rho\sin\rho\pi + \omega\cos\rho\pi + \kappa(\rho), \tag{2.2.51}$$

where

$$\kappa(\rho) = \frac{1}{2}\int_0^\pi q(t)\cos\rho(\pi-2t)\,dt + O\Big(\frac{1}{\rho}\exp(|\tau|\pi)\Big).$$

Denote

$$G_\delta = \{\rho : |\rho - k| \ge \delta,\ k = 0, \pm 1, \pm 2, \ldots\},\ \delta > 0.$$

Let us show that

$$|\sin\rho\pi| \ge C_\delta \exp(|\tau|\pi), \quad \rho \in G_\delta, \tag{2.2.52}$$

$$|\Delta(\lambda)| \ge C_\delta |\rho| \exp(|\tau|\pi), \quad \rho \in G_\delta, |\rho| \ge \rho^*, \tag{2.2.53}$$

for sufficiently large $\rho^* = \rho^*(\delta)$.

Let $\rho = \sigma + i\tau$. It is sufficient to prove (2.2.52) for the domain

$$D_\delta = \{\rho :\ \sigma \in \Big[-\frac{1}{2}, \frac{1}{2}\Big],\ \tau \ge 0,\ |\rho| \ge \delta\}.$$

Denote

$$\theta(\rho) = |\sin\rho\pi| \exp(-|\tau|\pi).$$

Let $\rho \in D_\delta$. For $\tau \le 1, \quad \theta(\rho) \ge C_\delta$. Since

$$\sin\rho\pi = \frac{\exp(i\rho\pi) - \exp(-i\rho\pi)}{2i},$$

we have for $\tau \geq 1$,

$$\theta(\rho) = \frac{1}{2}|1 - \exp(2i\sigma\pi)\exp(-2\tau\pi)| \geq \frac{1}{4}.$$

Thus, (2.2.52) is proved. Further, using (2.2.51) we get for $\rho \in G_\delta$,

$$\Delta(\lambda) = -\rho\sin\rho\pi\left(1 + O\left(\frac{1}{\rho}\right)\right),$$

and consequently (2.2.53) is valid.

3) Denote

$$\Gamma_n = \{\lambda : |\lambda| = (n+1/2)^2\}.$$

By virtue of (2.2.51),

$$\Delta(\lambda) = f(\lambda) + g(\lambda), \quad f(\lambda) = -\rho\sin\rho\pi, \quad |g(\lambda)| \leq C\exp(|\tau|\pi).$$

According to (2.2.52), $|f(\lambda)| > |g(\lambda)|, \quad \lambda \in \Gamma_n,$ for sufficiently large n $(n \geq n^*)$. Then by Rouchè's theorem, the number of zeros of $\Delta(\lambda)$ inside Γ_n coincides with the number of zeros of $f(\lambda) = -\rho\sin\rho\pi,$ i.e. it equals $n+1$. Thus, in the circle $|\lambda| < (n+1/2)^2$ there exist exactly $n+1$ eigenvalues of $L:\ \lambda_0, \dots, \lambda_n$. Applying now Rouchè's theorem to the circle $\gamma_n(\delta) = \{\rho :\ |\rho - n| \leq \delta\},$ we conclude that for sufficiently large n, in $\gamma_n(\delta)$ there is exactly one zero of $\Delta(\rho^2)$, namely $\rho_n = \sqrt{\lambda_n}$. Since $\delta > 0$ is arbitrary, it follows that

$$\rho_n = n + \varepsilon_n, \quad \varepsilon_n = o(1),\ n \to \infty. \tag{2.2.54}$$

Substituting (2.2.54) into (2.2.51) we get

$$0 = \Delta(\rho_n^2) = -(n+\varepsilon_n)\sin(n+\varepsilon_n)\pi + \omega\cos(n+\varepsilon_n)\pi + \kappa_n,$$

and consequently

$$-n\sin\varepsilon_n\pi + \omega\cos\varepsilon_n\pi + \kappa_n = 0. \tag{2.2.55}$$

Then

$$\sin\varepsilon_n\pi = O\left(\frac{1}{n}\right),$$

i.e.

$$\varepsilon_n = O\left(\frac{1}{n}\right).$$

Using (2.2.55) once more we obtain more precisely

$$\varepsilon_n = \frac{\omega}{\pi n} + \frac{\kappa_n}{n},$$

i.e. (2.2.49) is valid. Using (2.2.49) we arrive at (2.2.50), where

$$\xi_n(x) = \left(h + \frac{1}{2}\int_0^x q(t)\,dt - x\frac{\omega}{\pi} - x\kappa_n\right)\sin nx$$
$$+\frac{1}{2}\int_0^x q(t)\sin n(x-2t)\,dt + O\Big(\frac{1}{n}\Big).$$

Consequently $|\xi_n(x)| \le C,$ and Theorem 2.2.5 is proved. □

By virtue of (2.2.42) with $x = \pi,$

$$\beta_n = (\varphi(\pi, \lambda_n))^{-1}.$$

Then, using (2.2.43), (2.2.44) and (2.2.50) one can calculate

$$\alpha_n = \frac{\pi}{2} + \frac{\kappa_n}{n}, \quad \beta_n = (-1)^n + \frac{\kappa_n}{n}. \tag{2.2.56}$$

Since $\Delta(\lambda)$ has only simple zeros, we have

$$\operatorname{sign} \dot{\Delta}(\lambda_n) = (-1)^{n+1}$$

for $n \ge 0.$

Theorem 2.2.6. *The specification of the spectrum* $\{\lambda_n\}_{n\ge 0}$ *uniquely determines the characteristic function* $\Delta(\lambda)$ *by the formula*

$$\Delta(\lambda) = \pi(\lambda_0 - \lambda) \prod_{n=1}^{\infty} \frac{\lambda_n - \lambda}{n^2}. \tag{2.2.57}$$

Proof. It follows from (2.2.51) that $\Delta(\lambda)$ is entire in λ of order 1/2, and consequently by Hadamard‘s factorization theorem, $\Delta(\lambda)$ is uniquely determined up to a multiplicative constant by its zeros:

$$\Delta(\lambda) = C \prod_{n=0}^{\infty} \left(1 - \frac{\lambda}{\lambda_n}\right) \tag{2.2.58}$$

(the case when $\Delta(0) = 0$ requires minor modifications). Consider the function

$$\tilde{\Delta}(\lambda) := -\rho \sin \rho\pi = -\lambda\pi \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{n^2}\right).$$

Then

$$\frac{\Delta(\lambda)}{\tilde{\Delta}(\lambda)} = C \frac{\lambda - \lambda_0}{\lambda_0 \pi \lambda} \prod_{n=1}^{\infty} \frac{n^2}{\lambda_n} \prod_{n=1}^{\infty} \left(1 + \frac{\lambda_n - n^2}{n^2 - \lambda}\right).$$

Taking (2.2.49) and (2.2.51) into account we calculate

$$\lim_{\lambda \to -\infty} \frac{\Delta(\lambda)}{\tilde{\Delta}(\lambda)} = 1,$$

$$\lim_{\lambda \to -\infty} \prod_{n=1}^{\infty} \left(1 + \frac{\lambda_n - n^2}{n^2 - \lambda}\right) = 1,$$

and hence

$$C = \pi\lambda_0 \prod_{n=1}^{\infty} \frac{\lambda_n}{n^2}.$$

Substituting this into (2.2.58) we arrive at (2.2.57). □

Now we are going to prove that the system of the eigenfunctions of the Sturm-Liouville boundary value problem L is complete and forms an orthogonal basis in $L_2(0,\pi)$. This theorem was first proved by Steklov at the end of XIX-th century. We also provide sufficient conditions under which the Fourier series for the eigenfunctions converges uniformly on $[0,\pi]$. The completeness and expansion theorems are important for solving various problems in mathematical physics by the Fourier method, and also for the spectral theory itself. In order to prove these theorems we apply the contour integral method of integrating the resolvent along expanding contours in the complex plane of the spectral parameter (since this method is based on Cauchy‘s theorem, it sometimes called Cauchy‘s method).

Theorem 2.2.7. (i) *The system of eigenfunctions* $\{\varphi(x,\lambda_n)\}_{n\geq 0}$ *of the boundary value problem* L *is complete in* $L_2(0,\pi)$.

(ii) *Let* $f(x)$, $x\in[0,\pi]$ *be an absolutely continuous function. Then*

$$f(x)=\sum_{n=0}^{\infty}a_n\varphi(x,\lambda_n),\quad a_n=\frac{1}{\alpha_n}\int_0^{\pi}f(t)\varphi(t,\lambda_n)\,dt, \tag{2.2.59}$$

and the series converges uniformly on $[0,\pi]$.

(iii) *For* $f(x)\in L_2(0,\pi)$, *the series* (2.2.59) *converges in* $L_2(0,\pi)$, *and*

$$\int_0^{\pi}|f(x)|^2\,dx=\sum_{n=0}^{\infty}\alpha_n|a_n|^2\quad\text{(Parseval‘s equality)}\,. \tag{2.2.60}$$

Proof. 1) Denote

$$G(x,t,\lambda)=-\frac{1}{\Delta(\lambda)}\begin{cases}\varphi(x,\lambda)\psi(t,\lambda), & x\leq t,\\ \varphi(t,\lambda)\psi(x,\lambda), & x\geq t,\end{cases}$$

and consider the function

$$Y(x,\lambda)=\int_0^{\pi}G(x,t,\lambda)f(t)\,dt$$
$$=-\frac{1}{\Delta(\lambda)}\Big(\psi(x,\lambda)\int_0^x\varphi(t,\lambda)f(t)\,dt+\varphi(x,\lambda)\int_x^{\pi}\psi(t,\lambda)f(t)\,dt\Big).$$

The function $G(x,t,\lambda)$ is called *Green‘s function* for L. $G(x,t,\lambda)$ is the kernel of the inverse operator for the Sturm-Liouville operator, i.e. $Y(x,\lambda)$ is the solution of the boundary value problem

$$\left.\begin{aligned}\ell Y-\lambda Y+f(x)=0,\\ U(Y)=V(Y)=0;\end{aligned}\right\} \tag{2.2.61}$$

this is easily verified by differentiation. Taking (2.2.42) into account and using Theorem 2.2.4 we calculate

$$\operatorname*{Res}_{\lambda=\lambda_n}Y(x,\lambda)=-\frac{1}{\dot{\Delta}(\lambda_n)}\Big(\psi(x,\lambda_n)\int_0^x\varphi(t,\lambda_n)f(t)\,dt$$
$$+\varphi(x,\lambda_n)\int_x^{\pi}\psi(t,\lambda_n)f(t)\,dt\Big),$$

and consequently,

$$\operatorname*{Res}_{\lambda=\lambda_n} Y(x,\lambda) = -\frac{\beta_n}{\dot{\Delta}(\lambda_n)}\varphi(x,\lambda_n)\int_0^\pi f(t)\varphi(t,\lambda_n)\,dt.$$

By virtue of (2.2.44),

$$\operatorname*{Res}_{\lambda=\lambda_n} Y(x,\lambda) = \frac{1}{\alpha_n}\varphi(x,\lambda_n)\int_0^\pi f(t)\varphi(t,\lambda_n)\,dt. \tag{2.2.62}$$

2) Let $f(x)\in L_2(0,\pi)$ be such that

$$\int_0^\pi f(t)\varphi(t,\lambda_n)\,dt = 0, \quad n\geq 0.$$

Then in view of (2.2.62),

$$\operatorname*{Res}_{\lambda=\lambda_n} Y(x,\lambda) = 0,$$

and consequently (after extending $Y(x,\lambda)$ continuously to the whole λ - plane) for each fixed $x\in[0,\pi]$, the function $Y(x,\lambda)$ is entire in λ. Furthermore, it follows from (2.2.45), (2.2.46) and (2.2.53) that for a fixed $\delta>0$ and sufficiently large $\rho^*>0$:

$$|Y(x,\lambda)|\leq \frac{C_\delta}{|\rho|}, \quad \rho\in G_\delta,\ |\rho|\geq\rho^*.$$

Using the maximum principle and Liouville‘s theorem we conclude that $Y(x,\lambda)\equiv 0$. From this and (2.2.61) it follows that $f(x)=0$ a.e. on $(0,\pi)$. Thus (i) is proved.

3) Let now $f\in AC[0,\pi]$ be an arbitrary absolutely continuous function. Since $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ are solutions of (2.2.37), we transform $Y(x,\lambda)$ as follows

$$Y(x,\lambda) = -\frac{1}{\lambda\Delta(\lambda)}\Big(\psi(x,\lambda)\int_0^x(-\varphi''(t,\lambda)+q(t)\varphi(t,\lambda))f(t)\,dt$$

$$+\varphi(x,\lambda)\int_x^\pi(-\psi''(t,\lambda)+q(t)\psi(t,\lambda))f(t)\,dt\Big).$$

Integration of the terms containing second derivatives by parts yields in view of (2.2.40),

$$Y(x,\lambda) = \frac{f(x)}{\lambda}-\frac{1}{\lambda}\Big(Z_1(x,\lambda)+Z_2(x,\lambda)\Big), \tag{2.2.63}$$

where

$$Z_1(x,\lambda) = \frac{1}{\Delta(\lambda)}\Big(\psi(x,\lambda)\int_0^x g(t)\varphi'(t,\lambda)\,dt$$

$$+\varphi(x,\lambda)\int_x^\pi g(t)\psi'(t,\lambda)\,dt\Big), \quad g(t):=f'(t),$$

$$Z_2(x,\lambda) = \frac{1}{\Delta(\lambda)}\Big(hf(0)\psi(x,\lambda)+Hf(\pi)\varphi(x,\lambda)$$

$$+\psi(x,\lambda)\int_0^x q(t)\varphi(t,\lambda)f(t))\,dt+\varphi(x,\lambda)\int_x^\pi q(t)\psi(t,\lambda)f(t))\,dt\Big).$$

Using (2.2.45), (2.2.46) and (2.2.53), we get for a fixed $\delta>0,$ and sufficiently large $\rho^*>0:$

$$\max_{0\le x\le\pi}|Z_2(x,\lambda)|\le\frac{C}{|\rho|},\quad \rho\in G_\delta,\quad |\rho|\ge\rho^*. \tag{2.2.64}$$

Let us show that

$$\lim_{\substack{|\rho|\to\infty\\ \rho\in G_\delta}}\max_{0\le x\le\pi}|Z_1(x,\lambda)|=0. \tag{2.2.65}$$

First we assume that $g(x)$ is absolutely continuous on $[0,\pi]$. In this case another integration by parts yields

$$Z_1(x,\lambda)=\frac{1}{\Delta(\lambda)}\Big(\psi(x,\lambda)g(t)\varphi(t,\lambda)\Big|_0^x+\varphi(x,\lambda)g(t)\psi(t,\lambda)\Big|_x^\pi$$

$$-\psi(x,\lambda)\int_0^x g'(t)\varphi(t,\lambda)\,dt-\varphi(x,\lambda)\int_x^\pi g'(t)\psi(t,\lambda)\,dt\Big).$$

By virtue of (2.2.45), (2.2.46) and (2.2.53), we infer

$$\max_{0\le x\le\pi}|Z_1(x,\lambda)|\le\frac{C}{|\rho|},\quad \rho\in G_\delta,\quad |\rho|\ge\rho^*.$$

Let now $g(t)\in L(0,\pi)$. Fix $\varepsilon>0$ and choose an absolutely continuous function $g_\varepsilon(t)$ such that

$$\int_0^\pi|g(t)-g_\varepsilon(t)|\,dt<\frac{\varepsilon}{2C^+},$$

where

$$C^+=\max_{0\le x\le\pi}\sup_{\rho\in G_\delta}\frac{1}{|\Delta(\lambda)|}\Big(|\psi(x,\lambda)|\int_0^x|\varphi'(t,\lambda)|\,dt$$

$$+|\varphi(x,\lambda)|\int_x^\pi|\psi'(t,\lambda)|\,dt\Big).$$

Then, for $\rho\in G_\delta,\ |\rho|\ge\rho^*,$ we have

$$\max_{0\le x\le\pi}|Z_1(x,\lambda)|\le\max_{0\le x\le\pi}|Z_1(x,\lambda;g_\varepsilon)|$$

$$+\max_{0\le x\le\pi}|Z_1(x,\lambda;g-g_\varepsilon)|\le\frac{\varepsilon}{2}+\frac{C(\varepsilon)}{|\rho|}.$$

Hence, there exists $\rho^0>0$ such that

$$\max_{0\le x\le\pi}|Z_1(x,\lambda)|\le\varepsilon$$

for $|\rho|>\rho^0$. By virtue of the arbitrariness of $\varepsilon>0,$ we arrive at (2.2.65).

Consider the contour integral

$$I_N(x)=\frac{1}{2\pi i}\int_{\Gamma_N}Y(x,\lambda)\,d\lambda,$$

where $\Gamma_N = \{\lambda : \ |\lambda| = (N+1/2)^2\}$ (with counterclockwise circuit). It follows from (2.2.63)-(2.2.65) that

$$I_N(x) = f(x) + \varepsilon_N(x), \quad \lim_{N\to\infty} \max_{0\le x\le\pi} |\varepsilon_N(x)| = 0. \tag{2.2.66}$$

On the other hand, we can calculate $I_N(x)$ with the help of the residue theorem. By virtue of (2.2.62),

$$I_N(x) = \sum_{n=0}^{N} a_n \varphi(x,\lambda_n),$$

where

$$a_n = \frac{1}{\alpha_n}\int_0^{\pi} f(t)\varphi(t,\lambda_n)\,dt.$$

Comparing this with (2.2.66) we arrive at (2.2.59), where the series converges uniformly on $[0,\pi]$, i.e. (ii) is proved.

4) Since the eigenfunctions $\{\varphi(x,\lambda_n)\}_{n\ge 0}$ are complete and orthogonal in $L_2(0,\pi)$, they form an orthogonal basis in $L_2(0,\pi)$, and Parseval‘s equality (2.2.60) is valid. Theorem 2.2.7 is proved. □

We note that the assertions of Theorem 2.2.7 remain valid also if $h_1 = 0$ and/or $H_1 = 0$ provided $f(0) = 0$ and/or $f(\pi) = 0$.

Let us show that the Sturm-Liouville problem (2.2.34)-(2.2.35) can be reduced to the case (2.2.36).

Step 1. It follows from (2.2.34) that

$$-k(x)Y''(x) - k'(x)Y'(x) + q(x)Y(x) = \lambda\rho(x)Y(x),$$

and consequently,

$$-Y''(x) + p(x)Y'(x) + q_0(x)Y(x) = \lambda r(x)Y(x), \quad 0 < x < l, \tag{2.2.67}$$

where

$$p(x) = -\frac{k'(x)}{k(x)}, \quad q_0(x) = \frac{q(x)}{k(x)}, \quad r(x) = \frac{\rho(x)}{k(x)} > 0.$$

We make the substitution

$$Y(x) = \exp\left(\frac{1}{2}\int_0^x p(s)\,ds\right) Z(x). \tag{2.2.68}$$

Differentiating (2.2.68) we calculate

$$Y'(x) = \exp\left(\frac{1}{2}\int_0^x p(s)\,ds\right)\left(Z'(x) + \frac{p(x)}{2}Z(x)\right),$$

$$Y''(x) = \exp\left(\frac{1}{2}\int_0^x p(s)\,ds\right)$$

$$\times\left(Z''(x) + p(x)Z'(x) + \left(\frac{p'(x)}{2} + \frac{p^2(x)}{4}\right)Z(x)\right).$$

Therefore, equation (2.2.67) is transformed to the following equation with respect to $Z(x)$:

$$-Z''(x)+a(x)Y(x)=\lambda r(x)Z(x), \quad 0<x<l, \tag{2.2.69}$$

where

$$a(x)=q_0(x)-\frac{p'(x)}{2}+\frac{p^2(x)}{4}.$$

The boundary conditions (2.2.35) takes the form

$$\left.\begin{aligned} h_1Z'(0)-h_0Z(0)=0,\\ H_1Z'(l)+H_0Z(l)=0, \end{aligned}\right\} \tag{2.2.70}$$

where

$$h_0=h-\frac{h_1p(0)}{2}, \quad H_0=H+\frac{H_1p(l)}{2}.$$

Step 2. Denote

$$R(x)=\sqrt{r(x)}>0, \quad T=\int_0^l R(\tau)\,d\tau.$$

We make the replacement

$$\xi=\int_0^x R(\tau)\,d\tau, \quad z(\xi)=\sqrt{R(x)}Z(x). \tag{2.2.71}$$

It follows from (2.2.71) that $\dfrac{d}{d\xi}=\dfrac{1}{R(x)}\dfrac{d}{dx}$. Hence

$$\frac{dz}{d\xi}=\frac{1}{\sqrt{R(x)}}Z'(x)+\frac{R'(x)}{2R(x)\sqrt{R(x)}}Z(x),$$

$$\frac{d^2z}{d\xi^2}=\frac{1}{R(x)\sqrt{R(x)}}Z''(x)+\Big(\frac{R''(x)}{2R^2(x)\sqrt{R(x)}}-\frac{3(R'(x))^2}{4R^3(x)\sqrt{R(x)}}\Big)Z(x).$$

Therefore, the boundary value problem (2.2.69)-(2.2.70) is transformed to the following boundary value problem with respect to $z(\xi)$:

$$-\frac{d^2z(\xi)}{d\xi^2}+P(\xi)z(\xi)=\lambda z(\xi), \quad 0<\xi<T, \tag{2.2.72}$$

$$z'(0)-\alpha z(0)=0, \quad z'(T)+\beta z(T)=0, \tag{2.2.73}$$

where

$$P(\xi)=\frac{R''(x)}{2R^3(x)}-\frac{3(R'(x))^2}{4R^4(x)}+\frac{a(x)}{R^2(x)}$$

$$=\frac{r''(x)}{4r^2(x)}-\frac{5(r'(x))^2}{16r^3(x)}+\frac{a(x)}{r(x)},$$

$$\alpha=\frac{h_0}{h_1R(0)}+\frac{R'(0)}{2R^2(0)}, \quad \beta=\frac{H_0}{H_1R(l)}-\frac{R'(l)}{2R^2(l)}.$$

Step 3. We make the substitution

$$\eta = \frac{\pi\xi}{T}, \quad y(\eta) = z\left(\frac{T}{\pi}\eta\right).$$

Then the boundary value problem (2.2.72)-(2.2.73) is transformed to the following boundary value problem with respect to $y(\eta)$:

$$-\frac{d^2y(\eta)}{d\eta^2} + Q(\eta)y(\eta) = \mu y(\eta), \quad 0 < \eta < \pi,$$

$$y'(0) - \alpha_0 y(0) = 0, \quad y'(\pi) + \beta_0 y(T) = 0,$$

where

$$Q(\eta) = \frac{T^2}{\pi^2}P\left(\frac{T}{\pi}\eta\right), \quad \mu = \frac{T^2}{\pi^2}\lambda,$$

$$\alpha_0 = \frac{T}{\pi}\alpha, \quad \beta_0 = \frac{T}{\pi}\beta.$$

Thus, we have reduced the Sturm-Liouville problem (2.2.34)-(2.2.35) to the form (2.2.36). Using the results obtained above we conclude that the following theorem holds.

Theorem 2.2.8. (1) *The boundary value problem* $(2.2.34)-(2.2.35)$ *has a countable set of eigenvalues* $\{\lambda_n\}_{n\geq 0}$. *They are real and simple* (*i.e.* $\lambda_n \neq \lambda_k$ *for* $n \neq k$) , *and*

$$\rho_n := \sqrt{\lambda_n} = \frac{n\pi}{T} + O\left(\frac{1}{n}\right), \quad n \to \infty, \tag{2.2.74}$$

where

$$T = \int_0^l \sqrt{\frac{\rho(\tau)}{k(\tau)}}\,d\tau.$$

For each eigenvalue λ_n , *there exists only one* (*up to a multiplicative constant*) *eigenfunction* $Y_n(x)$. *For* $n \to \infty$,

$$Y_n(x) = \frac{1}{(k(x)\rho(x))^{1/4}}\cos\left(\frac{\pi n}{T}\int_0^x \sqrt{\frac{\rho(s)}{k(s)}}\,ds\right) + O\left(\frac{1}{n}\right)$$

uniformly in $x \in [0,l]$.

(2) *The eigenfunctions related to different eigenvalues are orthogonal in* $L_{2,\rho}(0,l)$, *i.e.*

$$\int_0^l Y_n(x)Y_k(x)\rho(x)\,dx = 0 \quad \text{for} \quad n \neq k.$$

The system of eigenfunctions $\{Y_n(x)\}_{n\geq 0}$ *is complete in* $L_{2,\rho}(0,l)$.

(3) *Let* $f(x),\ x \in [0,l]$ *be an absolutely continuous function. Then*

$$f(x) = \sum_{n=0}^{\infty} a_n Y_n(x),$$

where

$$a_n = \frac{1}{\alpha_n}\int_0^l f(x)Y_n(x)\rho(x)\,dx, \quad \alpha_n = \int_0^l Y_n^2(x)\rho(x)\,dx.$$

and the series converges uniformly on $[0,l]$.

4.3. Since nontrivial solutions of problem (2.2.34)-(2.2.35) exist only for the eigenvalues $\lambda=\lambda_n$ of the form (2.2.74), it makes sense to consider equation (2.2.33) only for $\lambda=\lambda_n$:

$$\ddot{T}_n(t)+\rho_n^2 T_n(t)=0, \quad n\ge 0, \quad \lambda_n=\rho_n^2.$$

The general solution of this equation has the form

$$T_n(t)=A_n\frac{\sin\rho_n t}{\rho_n}+B_n\cos\rho_n t, \quad n\ge 0,$$

where A_n and B_n are arbitrary constants. Thus, the solutions of the auxiliary problem have the form

$$u_n(x,t)=\left(A_n\frac{\sin\rho_n t}{\rho_n}+B_n\cos\rho_n t\right)Y_n(x), \quad n=0,1,2,\dots \tag{2.2.75}$$

We will seek the solution of the mixed problem (2.2.27)-(2.2.29) by superposition of standing waves (2.2.75), i.e. as a formal series solution:

$$u(x,t)=\sum_{n=0}^{\infty}\left(A_n\frac{\sin\rho_n t}{\rho_n}+B_n\cos\rho_n t\right)Y_n(x). \tag{2.2.76}$$

By construction the function $u(x,t)$ satisfies formally equation (2.2.27) and the boundary conditions (2.2.28) for all A_n and B_n. Choose A_n and B_n such that $u(x,t)$ satisfies also the initial conditions (2.2.29). Substituting (2.2.76) into (2.2.29) we calculate

$$\Phi(x)=\sum_{n=0}^{\infty}B_nY_n(x), \quad \Psi(x)=\sum_{n=0}^{\infty}A_nY_n(x),$$

and consequently,

$$\left.\begin{aligned} B_n&=\frac{1}{\alpha_n}\int_0^l \Phi(x)Y_n(x)\rho(x)\,dx,\\ A_n&=\frac{1}{\alpha_n}\int_0^l \Psi(x)Y_n(x)\rho(x)\,dx.\end{aligned}\right\} \tag{2.2.77}$$

Thus, the solution of the mixed problem (2.2.27)-(2.2.29) has the form (2.2.76), where the coefficients A_n and B_n is taken from (2.2.77).

We note that the series in (2.2.76) converges uniformly in $D:=\{(x,t):\ 0\le x\le l, t\ge 0\}$. Indeed, let

$$\ell_0 Y(x):=-(k(x)Y'(x))'+q(x)Y(x).$$

Since $\ell_0 Y_n(x) = \lambda_n \rho(x) Y_n(x)$, it follows from (2.2.77) that

$$A_n = \frac{1}{\alpha_n \lambda_n} \int_0^l \Psi(x) \ell_0 Y_n(x)\, dx$$

$$= -\frac{1}{\alpha_n \lambda_n} \int_0^l \Psi(x)(k(x)Y_n'(x))'\, dx + \frac{1}{\alpha_n \lambda_n} \int_0^l \Psi(x) q(x) Y_n(x)\, dx.$$

Integrating by parts twice the first integral we get

$$A_n = \frac{1}{\alpha_n \lambda_n} \left(k(x)(\Psi'(x)Y_n(x) - \Psi(x)Y_n'(x)) \right)\Big|_0^l$$

$$-\frac{1}{\alpha_n \lambda_n} \int_0^l (k(x)\Psi'(x))' Y_n(x)\, dx + \frac{1}{\alpha_n \lambda_n} \int_0^l \Psi(x) q(x) Y_n(x)\, dx.$$

The substitution vanishes since the functions $\Psi(x)$ and $Y_n(x)$ satisfy the boundary conditions (2.2.35). Therefore,

$$A_n = \frac{1}{\alpha_n \lambda_n} \int_0^l \Psi_0(x) Y_n(x)\, dx,$$

where $\Psi_0(x) := \ell_0 \Psi(x)$. Similarly,

$$B_n = \frac{1}{\alpha_n \lambda_n} \int_0^l \Phi_0(x) Y_n(x)\, dx,$$

where $\Phi_0(x) := \ell_0 \Phi(x)$. Taking the asymptotics for λ_n and $Y_n(x)$ into account (see Theorem 2.2.8) we conclude that the series (2.2.76) converges uniformly in D.

2.3. The Goursat Problem

The *Goursat problem* is the problem of solving a hyperbolic equation with given data on characteristics. Therefore, this problem is also called the problem on characteristics. It is convenient for us to consider the canonical form of hyperbolic equations for which the characteristics are parallel to the coordinate axes. For simplicity, we confine ourselves to linear equations. So, we consider the following Goursat problem:

$$u_{xy} = a(x,y)u_x + b(x,y)u_y + c(x,y)u + f(x,y), \quad (x,y) \in \Pi, \tag{2.3.1}$$

$$u_{|x=x_0} = \varphi(y), \quad u_{|y=y_0} = \psi(x). \tag{2.3.2}$$

Here x, y are the independent variables, $u(x,y)$ is an unknown function and $\Pi = \{(x,y) : x_0 \le x \le x_1, y_0 \le y \le y_1\}$ is a rectangle (see fig. 2.3.1). Equation (2.3.1) has two families of characteristics $x = const$ and $y = const$. . Thus, the conditions (2.3.2) are conditions on the characteristics $x = x_0$ and $y = y_0$.

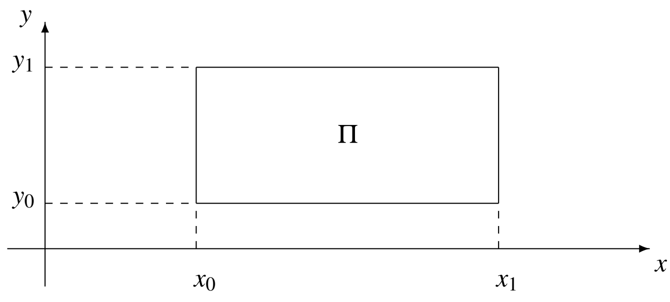

Figure 2.3.1.

Definition 2.3.1. The function $u(x,y)$ is called a solution of problem (2.3.1)-(2.3.2) if $u(x,y)$ is defined and continuous with its derivatives u_x, u_y, u_{xy} in the rectangle Π and satisfies (2.3.1) and (2.3.2).

Theorem 2.3.1. *Let the functions* a,b,c,f *be continuous in* Π, *and let the functions* φ *and* ψ *be continuously differentiable with* $\varphi(y_0)=\psi(x_0)$. *Then the solution of problem* $(2.3.1)-(2.3.2)$ *exists and is unique.*

Proof. 1) We reduce the Goursat problem (2.3.1)-(2.3.2) to an equivalent system of integral equations. Suppose that the solution $u(x,t)$ of problem (2.3.1)-(2.3.2) exists. Put

$$v=u_x, \quad w=u_y.$$

Then

$$v_y = av+bw+cu+f, \quad w_x = av+bw+cu+f, \quad u_y = w. \tag{2.3.3}$$

Integrating (2.3.3) and taking (2.3.2) into account we obtain

$$\left.\begin{aligned} v(x,y) &= \psi'(x) + \int_{y_0}^{y} (av+bw+cu+f)(x,\eta)\,d\eta, \\ w(x,y) &= \varphi'(y) + \int_{x_0}^{x} (av+bw+cu+f)(\xi,y)\,d\xi, \\ u(x,y) &= \psi(x) + \int_{y_0}^{y} w(x,\eta)\,d\eta. \end{aligned}\right\} \tag{2.3.4}$$

Thus, if u is a solution of problem (2.3.1)-(2.3.2), then the triple u,v,w is a solution of the system (2.3.4).

The inverse assertion is also valid. Indeed, let the triple u,v,w define a solution of the system (2.3.4). Differentiating (2.3.4) we deduce that the equalities (2.3.3) hold. Moreover,

$$u_x(x,y) = \psi'(x) + \int_{y_0}^{y} w_x(x,\eta)\,d\eta$$

$$= \psi'(x) + \int_{y_0}^{y} (av+bw+cu+f)(x,\eta)\,d\eta = v(x,y).$$

Thus $u_x = v$. Together with (2.3.3) this yields that the function $u(x,t)$ is a solution of equation (2.3.1). Furthermore, it follows from (2.3.4) for $x = x_0$ and $y = y_0$ that

$$u(x,y_0) = \psi(x),$$

$$u(x_0,y) = \psi(x_0) + \int_{y_0}^{y} w(x_0,\eta)\,d\eta = \psi(x_0) + \int_{y_0}^{y} \varphi'(\eta)\,d\eta = \varphi(y),$$

i.e. $u(x,y)$ satisfies the conditions (2.3.2). Thus, the Goursat problem (2.3.1)-(2.3.2) is equivalent to the system (2.3.4).

2) We will solve the system (2.3.4) by the method of successive approximations. Put

$$v_0(x,y) = \psi'(x) + \int_{y_0}^{y} f(x,\eta)\,d\eta,$$

$$w_0(x,y) = \varphi'(y) + \int_{x_0}^{x} f(\xi,y)\,d\xi,$$

$$u_0(x,y) = \psi(x),$$

$$\left.\begin{aligned} v_{n+1}(x,y) &= \int_{y_0}^{y} (av_n + bw_n + cu_n)(x,\eta)\,d\eta, \\ w_{n+1}(x,y) &= \int_{x_0}^{x} (av_n + bw_n + cu_n)(\xi,y)\,d\xi, \\ u_{n+1}(x,y) &= \int_{y_0}^{y} w_n(x,\eta)\,d\eta. \end{aligned}\right\} \qquad (2.3.5)$$

Take the constants $M \geq 0$ and $K \geq 1$ such that

$$|u_0|, |v_0|, |w_0| \leq M, \quad |a| + |b| + |c| \leq K.$$

Using (2.3.5), by induction one gets the estimates:

$$|v_n(x,y)|, |w_n(x,y)|, |u_n(x,y)| \leq MK^n \frac{(x+y-x_0-y_0)^n}{n!}. \qquad (2.3.6)$$

Indeed, for $n = 0$ these estimates are obvious. Fix $N \geq 0$ and suppose that estimates (2.3.6) are valid for $n = \overline{0,N}$. Then, using (2.3.5) and (2.3.6) we obtain

$$|v_{N+1}(x,y)| \leq \int_{y_0}^{y} MK^N (|a(x,\eta)| + |b(x,\eta)|$$

$$+ |c(x,\eta)|) \frac{(x+\eta-x_0-y_0)^N}{N!}\,d\eta,$$

and consequently,

$$|v_{N+1}(x,y)| \leq MK^{N+1} \int_{y_0}^{y} \frac{(x+\eta-x_0-y_0)^N}{N!}\,d\eta$$

$$= MK^{N+1} \left.\frac{(x+\eta-x_0-y_0)^{N+1}}{(N+1)!}\right|_{y_0}^{y} \leq MK^{N+1} \frac{(x+y-x_0-y_0)^{N+1}}{(N+1)!}.$$

For w_n and u_n arguments are similar.

By virtue of (2.3.6), the series

$$\left.\begin{aligned} u(x,y) &= \sum_{n=0}^{\infty} u_n(x,y), \\ v(x,y) &= \sum_{n=0}^{\infty} v_n(x,y), \\ w(x,y) &= \sum_{n=0}^{\infty} w_n(x,y) \end{aligned}\right\} \tag{2.3.7}$$

converge absolutely and uniformly in Π (since they are majorized by the convergent numerical series $M\sum_{n=0}^{\infty} K^n \frac{(x_1+y_1-x_0-y_0)^n}{n!}$), and

$$|v(x,y)|, |w(x,y)|, |u(x,y)| \le M\exp(K(x_1+y_1-x_0-y_0)). \tag{2.3.8}$$

Obviously, the triple $u, v, w,$ constructed by (2.3.7), solves system (2.3.4).

3) Let us prove the uniqueness. Let the triples (u, v, w) and $(\tilde{u}, \tilde{v}, \tilde{w})$ be solutions of system (2.3.4). Then the functions $u^* = u - \tilde{u}$, $v^* = v - \tilde{v}$, $w^* = w - \tilde{w}$ satisfy the homogeneous system

$$\left.\begin{aligned} v^*(x,y) &= \int_{y_0}^{y} (av^* + bw^* + cu^*)(x,\eta)\,d\eta, \\ w^*(x,y) &= \int_{x_0}^{x} (av^* + bw^* + cu^*)(\xi,y)\,d\xi, \\ u^*(x,y) &= \int_{y_0}^{y} w^*(x,\eta)\,d\eta. \end{aligned}\right\}$$

Since the functions u^*, v^*, w^* are continuous in $\Pi,$ there exists a constant $M_1 > 0$ such that $|u^*|, |v^*|, |w^*| \le M_1$. Repeating the previous arguments, by induction we obtain the estimate

$$|v^*(x,y)|, |w^*(x,y)|, |u^*(x,y)| \le M_1 K^n \frac{(x+y-x_0-y_0)^n}{n!}.$$

As $n \to \infty$ this yields $u^*(x,y) = v^*(x,y) = w^*(x,y) = 0.$ Theorem 2.3.1 is proved. □

Let us study the stability of the solution of the Goursat problem.

Definition 2.3.2. The solution of the Goursat problem is called stable if for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that if $|\varphi^{(\nu)}(y) - \tilde{\varphi}^{(\nu)}(y)| \le \delta$, $|\psi^{(\nu)}(x) - \tilde{\psi}^{(\nu)}(x)| \le \delta$, $\nu = 0, 1$, $x \in [x_0, x_1]$, $y \in [y_0, y_1]$, then $|u(x,y) - \tilde{u}(x,y)| \le \varepsilon$, $|u_x(x,y) - \tilde{u}_x(x,y)| \le \varepsilon$, $|u_y(x,y) - \tilde{u}_y(x,y)| \le \varepsilon$ for all $(x,y) \in \Pi.$ Here $\tilde{u}(x,y)$ is the solution of the Goursat problem for the data $\tilde{\varphi}, \tilde{\psi}.$

Let us show that the solution of problem (2.3.1)-(2.3.2) is stable. Indeed, denote $u^* = u - \tilde{u}$, $v^* = v - \tilde{v}$, $w^* = w - \tilde{w}$, $\varphi^* = \varphi - \tilde{\varphi}$, $\psi^* = \psi - \tilde{\psi}$, $f^* = 0.$ By virtue of (2.3.8),

$$|v^*(x,y)|, |w^*(x,y)|, |u^*(x,y)| \le M^* \exp(K(x_1+y_1-x_0-y_0)),$$

where

$$M^* = \max(\max_x |\psi^*(x)|, \max_x |\psi^{*,\prime}(x)|, \max_y |\varphi^{*,\prime}(y)|) \leq \delta.$$

Choosing $\delta = \varepsilon \exp(-K(x_1 + y_1 - x_0 - y_0))$ and using the relation $M^* \leq \delta$, we get $|u^*(x,y)|, |v^*(x,y)|, |w^*(x,y)| \leq \varepsilon$. Thus, the Goursat problem (2.3.1)-(2.3.2) is well-posed.

Now we reformulate the results obtained above for another canonical form of hyperbolic equations. We consider the following Goursat problem

$$u_{xx} - u_{tt} + a(x,t)u_x + b(x,t)u_t + c(x,t)u = f(x,t), \tag{2.3.9}$$

$$(x,t) \in \Delta(x_0,t_0),$$

$$u(x, x - x_0 + t_0) = \varphi(x), \quad u(x, -x + x_0 + t_0) = \psi(x), \tag{2.3.10}$$

where

$$\Delta(x_0,t_0) = \{(x,t)) : \ t - t_0 + x_0 \leq x \leq -t + t_0 + x_0, \ 0 \leq t \leq t_0\}$$

is the characteristic triangular (see fig. 2.4.1). Equation (2.3.9) has two families of characteristics $x + t = const$ and $x - t = const$. Thus, the conditions (2.3.10) are conditions on the characteristics $I_1 : \ t = x - x_0 + t_0$ and $I_2 : \ t = -x + x_0 + t_0$. Problem (2.3.9)-(2.310) can be reduced to a problem of the form (2.3.1)-(2.3.2) by the replacement:

$$x = \xi - \eta + t_0, \quad t = \xi + \eta - x_0$$

This yields

$$\xi = \frac{t + x}{2} + \frac{x_0 - t_0}{2}, \quad \eta = \frac{t - x}{2} + \frac{x_0 + t_0}{2}.$$

Indeed, denote

$$\tilde{u}(\xi,\eta) = u(\xi - \eta + t_0, \xi + \eta - x_0) = u(x,t).$$

Then $u_x = (\tilde{u}_\xi - \tilde{u}_\eta)/2$, $u_t = (\tilde{u}_\xi + \tilde{u}_\eta)/2$, $u_{xx} - u_{tt} = -\tilde{u}_{\xi\eta}$, and consequently, equation (2.3.9) takes the form

$$\tilde{u}_{\xi\eta} = \tilde{a}(\xi,\eta)\tilde{u}_\xi + \tilde{b}(\xi,\eta)\tilde{u}_\eta + \tilde{c}(\xi,\eta)\tilde{u} + \tilde{f}(\xi,\eta), \tag{2.3.11}$$

where

$$\tilde{a} = \frac{a + b}{2}, \ \tilde{b} = -\frac{a - b}{2}, \ \tilde{c} = c, \ \tilde{f} = -f.$$

The characteristics I_1 and I_2 of equation (2.3.9) are transformed into characteristics of equation (2.3.11) $\eta = t_0$ and $\xi = x_0$, respectively, and conditions (2.3.10) become

$$\tilde{u}(\xi, t_0) = \tilde{\varphi}(\xi), \quad \tilde{u}(x_0, \eta) = \tilde{\psi}(\eta),$$

where $\tilde{\varphi}(\xi) = \varphi(\xi)$, $\tilde{\psi}(\eta) = \psi(-\eta + x_0 + t_0)$. Therefore, the following theorem is a corollary of Theorem 2.3.1.

Theorem 2.3.2. *Let the functions* a, b, c, f *be continuous in* $\Delta(x_0,t_0)$, *and let the functions* φ *and* ψ *be continuously differentiable with* $\varphi(x_0) = \psi(x_0)$. *Then the solution of the Goursat problem* $(2.3.9) - (2.3.10)$ *exists and is unique.*

2.4. The Riemann Method

Riemann's method is a classical technique for solving the Cauchy problem for hyperbolic linear equations in two variables; in particular it provides information for domains of dependence and influence for solutions.

We consider the following Cauchy problem

$$u_{xx} - u_{tt} + a(x,t)u_x + b(x,t)u_t + c(x,t)u = f(x,t), \tag{2.4.1}$$

$$-\infty < x < \infty, \quad t > 0,$$

$$u_{|t=0} = \varphi(x), \quad u_{t|t=0} = \psi(x). \tag{2.4.2}$$

Denote $D = \{(x,t) : -\infty < x < \infty, t \geq 0\}$. Suppose that $a, b \in C^1(D)$, $c, f \in C(D)$, $\varphi \in C^2(\mathbf{R})$, $\psi \in C^1(\mathbf{R})$.

Definition 2.4.1. A function $u(x,t)$ is called a solution of problem (2.4.1)-(2.4.2) if $u(x,t) \in C^2(D)$, and $u(x,t)$ satisfies (2.4.1) and (2.4.2).

Derivation of the Riemann formula. Denote

$$\mathcal{L}u = u_{xx} - u_{tt} + au_x + bu_t + cu,$$

$$\mathcal{L}^*v = v_{xx} - v_{tt} - (av)_x - (bu)_t + cv.$$

Fix a point (x_0, t_0) and consider the characteristic triangular $\Delta(x_0,t_0) = \{(x,t) : t - t_0 + x_0 \leq x \leq -t + t_0 + x_0, 0 \leq t \leq t_0\}$ (see fig. 2.4.1) with vertices at the points $M = (x_0, t_0)$, $P = (x_0 - t_0, 0)$ and $Q = (x_0 + t_0, 0)$. The boundary $\partial\Delta$ of Δ consists of three segments: $\partial\Delta = I_1 \cup I_2 \cup I_3$, where

$$I_1 = \overline{MP} := \{(x,t) : t = x - x_0 + t_0, x_0 - t_0 \leq x \leq x_0\},$$

$$I_2 = \overline{QM} := \{(x,t) : t = -x + x_0 + t_0, x_0 \leq x \leq x_0 + t_0\},$$

$$I_3 = \overline{PQ} := \{(x,t) : t = 0, x_0 - t_0 \leq x \leq x_0 + t_0\}.$$

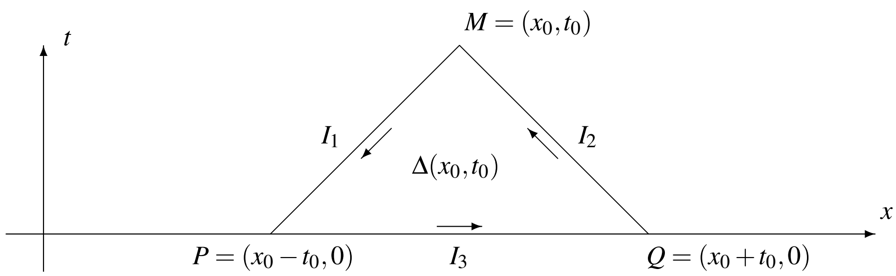

fig. 2.4.1

Suppose that the solution $u(x,t)$ of problem (2.4.1)-(2.4.2) exists. Let $v(x,t) \in C^2(D)$ be an arbitrary function. Since

$$v\mathcal{L}u - u\mathcal{L}^*v = (vu_x - uv_x + auv)_x - (vu_t - uv_t - buv)_t,$$

it follows from Green's formula that

$$\int_\Delta (v\mathcal{L}u - u\mathcal{L}^*v)\,dxdt$$

$$= \int_{\partial\Delta} (vu_t - uv_t - buv)\,dx + (vu_x - uv_x + auv)\,dt \tag{2.4.3}$$

(with counterclockwise orientation of $\partial\Delta$). Let us transform the integral along the boundary. We have

$$\int_{\partial\Delta} = \int_{I_1} + \int_{I_2} + \int_{I_3}.$$

1) On $I_1:\ t = x - x_0 + t_0,\ dt = dx.$ Denote

$$\alpha_1(x) = u(x, x - x_0 + t_0), \quad \beta_1(x) = v(x, x - x_0 + t_0).$$

Then $\alpha_1'(x) = (u_x + u_t)_{|t=x-x_0+t_0}$, $\beta_1'(x) = (v_x + v_t)_{|t=x-x_0+t_0}$, and consequently,

$$\int_{I_1} = \int_M^P (v(u_t + u_x) - u(v_t + v_x) + (a-b)uv)\,(x, x - x_0 + t_0)\,dx$$

$$= \int_M^P (\beta_1(x)\alpha_1'(x) - \alpha_1(x)\beta_1'(x)$$
$$+ (a-b)(x, x - x_0 + t_0)\alpha_1(x)\beta_1(x))\,dx.$$

Integrating by parts the first term we get

$$\int_{I_1} = \Big(\beta_1(x)\alpha_1(x)\Big)\Big|_M^P$$

$$-\int_M^P \alpha_1(x)\left(2\beta_1'(x) - (a-b)(x, x - x_0 + t_0)\beta_1(x)\right)\,dx.$$

Impose a first condition on the function $v(x,t)$, namely:

$$2\beta_1'(x) - (a-b)(x, x - x_0 + t_0)\beta_1(x) = 0.$$

Solving this ordinary differential equation we calculate

$$v(x, x - x_0 + t_0) = \exp\left(\frac{1}{2}\int_{x_0}^x (a-b)(\xi, \xi - x_0 + t_0)\,d\xi\right). \tag{2.4.4}$$

Then

$$\int_{I_1} = u(P)v(P) - u(M)v(M). \tag{2.4.5}$$

2) On $I_2: \ t=-x+x_0+t_0, dt=-dx.$ Denote

$$\alpha_2(x)=u(x,-x+x_0+t_0),\quad \beta_2(x)=v(x,-x+x_0+t_0).$$

Then $\alpha_2'(x)=(u_x-u_t)_{|t=-x+x_0+t_0},\ \beta_2'(x)=(v_x-v_t)_{|t=-x+x_0+t_0},$ and consequently,

$$\int_{I_2}=\int_Q^M (v(u_t-u_x)-u(v_t-v_x)-(a+b)uv)\,(x,-x+x_0+t_0)\,dx$$

$$=\int_Q^M(-\beta_2(x)\alpha_2'(x)+\alpha_2(x)\beta_2'(x)$$
$$-(a+b)(x,-x+x_0+t_0)\alpha_2(x)\beta_2(x))\,dx.$$

Integration by parts yields

$$\int_{I_2}=-\Big(\beta_2(x)\alpha_2(x)\Big)\Big|_Q^M$$
$$+\int_Q^M\alpha_2(x)\left(2\beta_2'(x)-(a+b)(x,-x+x_0+t_0)\beta_2(x)\right)\,dx.$$

Impose a second condition on the function $v(x,t),$ namely:

$$2\beta_2'(x)-(a+b)(x,-x+x_0+t_0)\beta_2(x)=0.$$

Solving this differential equation we calculate

$$v(x,-x+x_0+t_0)=\exp\left(\frac{1}{2}\int_{x_0}^x(a+b)(\xi,-\xi+x_0+t_0)\,d\xi\right). \tag{2.4.6}$$

Then

$$\int_{I_2}=u(Q)v(Q)-u(M)v(M). \tag{2.4.7}$$

3) On $I_3: \ t=0, dt=0,$ and consequently,

$$\int_{I_3}=\int_{x_0-t_0}^{x_0+t_0}(v(x,0)\psi(x)-v_t(x,0)\varphi(x)-b(x,0)v(x,0)\varphi(x))\,dx. \tag{2.4.8}$$

Impose a third condition on the function $v(x,t),$ namely:

$$\mathcal{L}^*v=0,\quad (x,t)\in\Delta. \tag{2.4.9}$$

Since (2.4.9), (2.4.4) and (2.4.6) is a Goursat problem, the function $v(x,t)$ exists and is unique, it is called the *Riemann function*. We note that $v(x,t)=v(x,t;x_0,t_0),$ i.e. the Riemann function depends on the point $(x_0,t_0).$ Substituting (2.4.5), (2.4.7), (2.4.8) and (2.4.9) into (2.4.3) and solving this with respect to $u(x_0,t_0),$ we obtain

$$u(x_0,t_0)=\frac{1}{2}\left(\varphi(x_0+t_0)v(x_0+t_0,0)+\varphi(x_0-t_0)v(x_0-t_0,0)\right)$$
$$+\frac{1}{2}\int_{x_0-t_0}^{x_0+t_0}(v(x,0)\psi(x)-v_t(x,0)\varphi(x)-b(x,0)v(x,0)\varphi(x))\,dx$$

$$-\frac{1}{2}\int_0^{t_0}\int_{x_0+t-t_0}^{x_0+t_0-t} v(x,t)f(x,t)\,dx\,dt. \tag{2.4.10}$$

Formula (2.4.10) is called the *Riemann formula*. Thus, we have proved that if the solution of problem (2.4.1)-(2.4.2) exists, then it is represented by formula (2.4.10). In particular, this yields the uniqueness of the solution of problem (2.4.1)-(2.4.2). One can verify (see, for example, [3, Ch.5]) that the function u, represented by (2.4.10), is really a solution of problem (2.4.1)-(2.4.2). We note that the existence of the solution of problem (2.4.1)-(2.4.2) can be proved independently by the method described in Section 2.3.

In some particular cases the Riemann function can be constructed explicitly.

Example 2.4.1. Consider the following Cauchy problem for the non-homogeneous equation of a vibrating string:

$$\left.\begin{aligned} u_{xx}-u_{tt}=f(x,t),\quad -\infty<x<\infty,\ t>0,\\ u_{|t=0}=\varphi(x),\quad u_{t|t=0}=\psi(x). \end{aligned}\right\} \tag{2.4.11}$$

This is the particular case of problem (2.4.1)-(2.4.2) when $a=b=c=0$. Clearly, in this case $v(x,t)\equiv 1$, and the Riemann formula (2.4.10) takes the form

$$u(x,t)=\frac{1}{2}\left(\varphi(x+t)+\varphi(x-t)\right)+\frac{1}{2}\int_{x-t}^{x+t}\psi(s)\,ds$$

$$-\frac{1}{2}\int_0^t\int_{x-t+\tau}^{x+t-\tau} f(s,\tau)\,ds\,d\tau. \tag{2.4.12}$$

Compare (2.4.12) with (2.1.11)!

Example 2.4.2. Consider the Cauchy problem (2.4.1)-(2.4.2) for the case, when the coefficients a,b and c are constants. Without loss of generality one can assume that $a=b=0$ (this can always be achieved by the replacement $u(x,t)=\tilde{u}(x,t)\exp(-ax/2+bt/2)$). Thus, we consider the following Cauchy problem

$$\left.\begin{aligned} u_{xx}-u_{tt}+cu=f(x,t),\quad -\infty<x<\infty,\ t>0;\quad c\neq 0,\\ u_{|t=0}=\varphi(x),\quad u_{t|t=0}=\psi(x). \end{aligned}\right\} \tag{2.4.13}$$

For the Riemann function we have the Goursat problem

$$\left.\begin{aligned} v_{xx}-v_{tt}+cv=0,\\ v_{|t=x-x_0+t_0}=1,\quad v_{|t=-x+x_0+t_0}=1. \end{aligned}\right\} \tag{2.4.14}$$

We seek a solution of (2.4.14) in the form

$$v(x,t)=w(z),\quad \text{where}\quad z=\sqrt{(t-t_0)^2-(x-x_0)^2}.$$

Clearly, if the point (x,t) lies on the characteristics I_1 or I_2, then $z=0$, moreover $z>0$ inside the triangular $\Delta(x_0,t_0)$. Since

$$\frac{\partial z}{\partial t}=\frac{t-t_0}{z},\quad \frac{\partial z}{\partial x}=-\frac{x-x_0}{z},$$

we have

$$v_x = -w' \frac{x-x_0}{z}, \quad v_{xx} = w'' \frac{(x-x_0)^2}{z^2} - w' \frac{1}{z} - w' \frac{(x-x_0)^2}{z^3},$$

$$v_t = w' \frac{t-t_0}{z}, \quad v_{tt} = w'' \frac{(t-t_0)^2}{z^2} + w' \frac{1}{z} - w' \frac{(t-t_0)^2}{z^3}.$$

Substituting this into (2.4.14) we obtain

$$w''(z) + \frac{w'(z)}{z} - cw(z) = 0, \quad w(0) = 1. \tag{2.4.15}$$

The replacement $\xi = \sqrt{-c}z,\ y(\xi) = w(z)$ in (2.4.15) yields

$$y''(\xi) + \frac{y'(\xi)}{\xi} + y(\xi) = 0, \quad y(0) = 1,$$

and consequently,

$$y(\xi) = J_0(\xi),$$

where

$$J_0(\xi) := 1 - \left(\frac{\xi}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{\xi}{2}\right)^4 - \frac{1}{(3!)^3}\left(\frac{\xi}{2}\right)^6 + \cdots$$

is the Bessel function of zero order [1, p.636]. Thus, the Riemann function has the form

$$v(x,t) = J_0\left(\sqrt{-c((t-t_0)^2 - (x-x_0)^2)}\right).$$

2.5. The Cauchy Problem for the Wave Equation

We consider the following Cauchy problem

$$u_{tt} = \Delta u, \quad x \in \mathbf{R}^3,\ t > 0, \tag{2.5.1}$$

$$u_{|t=0} = \varphi(x), \quad u_{t|t=0} = \psi(x). \tag{2.5.2}$$

Here $x = (x_1, x_2, x_3) \in \mathbf{R}^3,\ t \ge 0$ are independent variables, $u(x,t)$ is an unknown function and

$$\Delta u := \sum_{k=1}^{3} \frac{\partial^2 u}{\partial x_k^2}$$

is the Laplace operator. Denote $D = \{(x,t) :\ x \in \mathbf{R}^3,\ t \ge 0\}$.

Definition 2.5.1. A function $u(x,t)$ is called a solution of (2.5.1)-(2.5.2), if $u(x,t) \in C^2(D)$, and $u(x,t)$ satisfies (2.5.1)-(2.5.2).

Fix a point $x^0 = (x_1^0, x_2^0, x_3^0) \in \mathbf{R}^3$ and denote by

$$r = \|x - x^0\| = \sqrt{\sum_{k=1}^{3} (x_k - x_k^0)^2}$$

the distance between the points x and x_0. Let $K(x^0,\alpha) := \{x \in \mathbf{R}^3 : \; r \le \alpha\}$ be the closed ball of the radius α with center x_0, and let here and below $\partial K(x^0,\alpha) := \{x \in \mathbf{R}^3 : \; r = \alpha\}$ be the corresponding sphere (the boundary of the ball $K(x^0,\alpha)$).

Lemma 2.5.1. $\Delta\left(\frac{1}{r}\right) = 0.$

Proof. Since

$$\frac{\partial r}{\partial x_k} = \frac{x_k - x_k^0}{r},$$

we have

$$\frac{\partial}{\partial x_k}\left(\frac{1}{r}\right) = -\frac{x_k - x_k^0}{r^3},$$

$$\frac{\partial^2}{\partial x_k^2}\left(\frac{1}{r}\right) = -\frac{1}{r^3} + \frac{3(x_k - x_k^0)^2}{r^5},$$

and consequently, $\Delta\left(\frac{1}{r}\right) = 0.$ □

Derivation of Kirchhoff's formula. Fix a point $(x^0,t_0) \in \mathbf{R}^4$. In the four-dimensional space of variables (x,t) we consider the cone G with the vertex at the point (x^0,t_0), the generator $t_0 - t = r$ and the base $K(x^0,t_0)$, in the hyper-plane $t = 0$ (see fig. 2.5.1). Put

$$v(x,t) = \frac{t_0 - t}{r} - 1.$$

By virtue of Lemma 2.5.1, the function $v(x,t)$ satisfies equation (2.5.1):

$$v_{tt} = \Delta v.$$

Moreover, $v(x,t) \ge 0$ inside the cone G, and $v(x,t) = 0$ on the lateral surface of G. On the axis of the cone we have $x = x^0$, and the function $v(x,t)$ has a singularity.

Fix $\delta > 0$. Denote by

$$\Omega(x^0,\delta) = \{(x,t) \in \mathbf{R}^4 : \; r \le \delta, t \ge 0\}$$

the cylinder with the axis $x = x^0$ and of radius δ, and consider the domain $G_\delta = G \setminus \Omega(x^0,\delta)$ which is the cone without the cylinder (see fig. 2.5.1). The boundary S_δ of G_δ consists of three parts:

$$S_\delta = S_{1,\delta} \cup S_{2,\delta} \cup S_{3,\delta},$$

where

$$S_{1,\delta} = \{(x,t) : \; t = 0, \delta \le r \le t_0\},$$

$$S_{2,\delta} = \{(x,t) : \; t_0 - t = r, 0 \le t \le t_0 - \delta\}$$

is the lateral surface of the cone, and

$$S_{3,\delta} = \{(x,t) : \; r = \delta, 0 \le t \le t_0 - \delta\}$$

is the lateral surface of the cylinder.

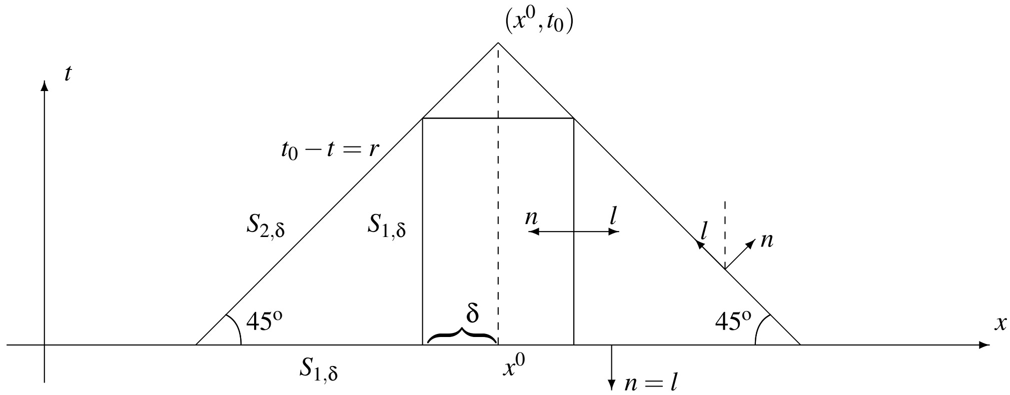

fig. 2.5.1

Suppose that the solution $u(x,t)$ of problem (2.5.1)-(2.5.2) exists. Since $u_{tt}-\Delta u$ and $v_{tt}-\Delta v$, it follows that

$$\int_{G_\delta} (u(v_{tt}-\Delta v)-v(u_{tt}-\Delta u))\,dxdt=0,$$

and consequently,

$$\int_{G_\delta}\left(\frac{\partial}{\partial t}(uv_t-vu_t)-\sum_{k=1}^{3}\frac{\partial}{\partial x_k}(uv_{x_k}-vu_{x_k})\right)dxdt=0.$$

Using the Gauß-Ostrogradskii formula , we calculate

$$\int_{G_\delta}\left(\frac{\partial}{\partial t}(uv_t-vu_t)-\sum_{k=1}^{3}\frac{\partial}{\partial x_k}(uv_{x_k}-vu_{x_k})\right)dxdt$$

$$=\int_{S_\delta}\left((uv_t-vu_t)\cos(n,t)-\sum_{k=1}^{3}(uv_{x_k}-vu_{x_k})\cos(n,x_k)\right)ds,$$

where n is the outer normal to the surface, and $\cos(n,t)$ and $\cos(n,x_k)$ are the cosines of the angles between the outer normal and the corresponding coordinate axes. Consequently,

$$\int_{S_\delta}\Big(\Big(v_t\cos(n,t)-\sum_{k=1}^{3}v_{x_k}\cos(n,x_k)\Big)u$$

$$-v\Big(u_t\cos(n,t)-\sum_{k=1}^{3}u_{x_k}\cos(n,x_k)\Big)\Big)\,ds=0.$$

Choose the direction l such that

$$\cos(l,t)=\cos(n,t),\quad \cos(l,x_k)=-\cos(n,x_k).$$

Then

$$\int_{S_\delta}\left(u\frac{\partial v}{\partial l}-v\frac{\partial u}{\partial l}\right)ds=0. \tag{2.5.3}$$

1) On $S_{1,\delta}:\ n = l,\ \dfrac{\partial}{\partial l} = -\dfrac{\partial}{\partial t},\ ds = dx,\ t = 0,$ and consequently,

$$\int_{S_{1,\delta}} \left(u\frac{\partial v}{\partial l} - v\frac{\partial u}{\partial l} \right) ds = \int_{S_{1,\delta}} \left(-u\frac{\partial v}{\partial t} + v\frac{\partial u}{\partial t} \right) dx$$

$$= \int_{S_{1,\delta}} \left(\frac{\varphi(x)}{r} + \left(\frac{t_0}{r} - 1 \right) \psi(x) \right) dx.$$

As $\delta \to 0$ we get

$$\lim_{\delta\to 0} \int_{S_{1,\delta}} \left(u\frac{\partial v}{\partial l} - v\frac{\partial u}{\partial l} \right) ds = \int_{K(x^0,t_0)} \left(\frac{\varphi(x)}{r} + \left(\frac{t_0}{r} - 1 \right) \psi(x) \right) dx. \tag{2.5.4}$$

2) On $S_{2,\delta}$ the direction l coincides with the direction of the generator of the cone, hence $v = \dfrac{\partial v}{\partial l} = 0$ on $S_{2,\delta}$. This yields

$$\int_{S_{2,\delta}} \left(u\frac{\partial v}{\partial l} - v\frac{\partial u}{\partial l} \right) ds = 0. \tag{2.5.5}$$

3) On $S_{3,\delta}:\ r = \delta.$ Then

$$\int_{S_{3,\delta}} v\frac{\partial u}{\partial l}\, ds = \int_0^{t_0-\delta} \left(\frac{t_0 - t}{\delta} - 1 \right) dt \int_{\partial K(x^0,\delta)} \frac{\partial u}{\partial l}\, ds.$$

Since $\left| \dfrac{\partial u}{\partial l} \right| \le C,$ we have

$$\left| \int_{S_{3,\delta}} v\frac{\partial u}{\partial l}\, ds \right| \le 4\pi\delta^2 C \int_0^{t_0-\delta} \left(\frac{t_0 - t}{\delta} - 1 \right) dt \le C_1\delta,$$

and consequently,

$$\lim_{\delta\to 0} \int_{S_{3,\delta}} v\frac{\partial u}{\partial l}\, ds = 0.$$

Furthermore, on $S_{3,\delta}:\ l = -n = \bar{r} := (x_1 - x_1^0, x_2 - x_2^0, x_3 - x_3^0).$ Therefore $\dfrac{\partial}{\partial l} = \dfrac{\partial v}{\partial r}$ and

$$\int_{S_{3,\delta}} u\frac{\partial v}{\partial l}\, ds = \int_{S_{3,\delta}} u\frac{\partial v}{\partial r}\, ds = \int_{S_{3,\delta}} \left(-\frac{t_0 - t}{r^2} \right) u\, ds$$

$$= -\frac{1}{\delta^2} \int_0^{t_0-\delta} (t_0 - t)\, dt \int_{\partial K(x^0,\delta)} u(x,t)\, ds$$

$$= -4\pi \int_0^{t_0-\delta} (t_0 - t) u(x^0,t)\, dt + J_\delta,$$

where

$$J_\delta = -\frac{1}{\delta^2} \int_0^{t_0-\delta} (t_0 - t)\, dt \int_{\partial K(x^0,\delta)} (u(x,t) - u(x^0,t))\, ds.$$

By virtue of the continuity of $u(x,t)$, for each $\varepsilon>0$ there exists $\delta>0$ such that if $r\le\delta$, then $|u(x,t)-u(x^0,t)|\le\varepsilon$ for all $t\in[0,t_0]$. Therefore,

$$|J_\delta|\le\frac{\varepsilon}{\delta^2}\int_0^{t_0-\delta}(t_0-t)\,dt\int_{\partial K(x^0,\delta)}ds\le 4\pi\varepsilon\int_0^{t_0}(t_0-t)\,dt.$$

Thus,

$$\lim_{\delta\to 0}\int_{S_{3,\delta}}\left(u\frac{\partial v}{\partial l}-v\frac{\partial u}{\partial l}\right)ds=-4\pi\int_0^{t_0}(t_0-t)u(x^0,t)\,dt. \tag{2.5.6}$$

It follows from (2.5.3)-(2.5.6) that

$$\int_0^{t_0}(t_0-t)u(x^0,t)\,dt=\frac{1}{4\pi}\int_{K(x^0,t_0)}\left(\frac{\varphi(x)}{r}+\left(\frac{t_0}{r}-1\right)\psi(x)\right)dx. \tag{2.5.7}$$

Differentiating (2.5.7) twice with respect to t_0, we calculate

$$u(x^0,t_0)=\frac{1}{4\pi}\frac{\partial^2}{\partial t_0^2}\left(\int_0^{t_0}dr\int_{\partial K(x^0,r)}\left(\frac{\varphi(x)}{r}+\left(\frac{t_0}{r}-1\right)\psi(x)\right)ds\right)$$

or

$$u(x^0,t_0)=\frac{1}{4\pi}\frac{\partial}{\partial t_0}\int_{\partial K(x^0,t_0)}\frac{\varphi(x)}{t_0}ds+\frac{1}{4\pi}\int_{\partial K(x^0,t_0)}\frac{\psi(x)}{t_0}ds. \tag{2.5.8}$$

Formula (2.5.8) is called *Kirchhoff's formula*. Thus, we have proved that if the solution of problem (2.5.1)-(2.5.2) exists, then it is represented by (2.5.8). In particular, this yields the uniqueness of the solution of problem (2.5.1)-(2.5.2).

Theorem 2.5.1. *Let* $\varphi(x)\in C^3(\mathbf{R}^3)$, $\psi(x)\in C^2(\mathbf{R}^3)$. *Then the solution of the Cauchy problem* $(2.5.1)-(2.5.2)$ *exists, is unique and is represented by formula* $(2.5.8)$.

Proof. It is sufficient to prove that the function $u(x,t)$, defined by (2.5.8), satisfies (2.5.1)-(2.5.2). For this purpose we consider the following auxiliary function

$$w(x^0,t_0)=\frac{1}{4\pi}\int_{\partial K(x^0,t_0)}\frac{f(x)}{t_0}ds, \tag{2.5.9}$$

where $f(x)$ is a sufficiently smooth function. In (2.5.9) we make the replacement $x_k=x_k^0+t_0\xi_k$, $k=\overline{1,3}$ (i.e. $\xi_k=\dfrac{x_k-x_k^0}{t_0}$). If x belongs to the sphere $\partial K(x^0,t_0)$, then ξ belongs to the sphere $\partial K(0,1)$, and $ds_x=t_0^2ds_\xi$. Consequently,

$$w(x^0,t_0)=\frac{t_0}{4\pi}\int_{\partial K(0,1)}f(x^0+t_0\xi)\,ds. \tag{2.5.10}$$

It follows from (2.5.10) that the function $w(x^0,t_0)$ has the same smoothness properties as $f(x)$, and

$$w_{|t_0=0}=0. \tag{2.5.11}$$

Differentiating (2.5.10) we get

$$\frac{\partial w}{\partial t_0}=\frac{1}{4\pi}\int_{\partial K(0,1)}f(x^0+t_0\xi)\,ds$$

$$+\frac{t_0}{4\pi}\int_{\partial K(0,1)}\left(\sum_{k=1}^{3}\frac{\partial f}{\partial x_k}(x^0+t_0\xi)\cdot\xi_k\right)ds. \tag{2.5.12}$$

In particular, (2.5.12) yields

$$\left.\frac{\partial w}{\partial t_0}\right|_{t_0=0}=f(x^0). \tag{2.5.13}$$

We note that in (2.5.12) $\xi_k=\cos(n,\xi_k)$. Therefore, applying the Gauß-Ostrogradskii formula and using (2.5.10) and the relation $\dfrac{\partial}{\partial\xi_k}=t_0\dfrac{\partial}{\partial x_k}$, we obtain

$$\frac{\partial w}{\partial t_0}=\frac{w(x^0,t_0)}{t_0}+\frac{t_0}{4\pi}\int_{K(0,1)}\left(\sum_{k=1}^{3}\frac{\partial}{\partial\xi_k}\left(\frac{\partial f}{\partial x_k}(x^0+t_0\xi)\right)\right)d\xi$$

$$=\frac{w(x^0,t_0)}{t_0}+\frac{t_0^2}{4\pi}\int_{K(0,1)}\left(\sum_{k=1}^{3}\frac{\partial^2 f}{\partial x_k^2}(x^0+t_0\xi)\right)d\xi$$

$$=\frac{w(x^0,t_0)}{t_0}+\frac{1}{4\pi t_0}\int_{K(x^0,t_0)}\left(\sum_{k=1}^{3}\frac{\partial^2 f}{\partial x_k^2}(x)\right)dx.$$

Thus,

$$\frac{\partial w}{\partial t_0}=\frac{w(x^0,t_0)}{t_0}+\frac{1}{4\pi t_0}Q, \tag{2.5.14}$$

where

$$Q:=\int_{K(x^0,t_0)}\left(\sum_{k=1}^{3}\frac{\partial^2 f}{\partial x_k^2}(x)\right)dx=\int_0^{t_0}dr\int_{\partial K(x^0,r)}\left(\sum_{k=1}^{3}\frac{\partial^2 f}{\partial x_k^2}(x)\right)ds.$$

Using (2.5.14) we calculate

$$\frac{\partial^2 w}{\partial t_0^2}=\frac{1}{t_0}\cdot\frac{\partial w}{\partial t_0}-\frac{w}{t_0^2}-\frac{1}{4\pi t_0^2}Q+\frac{1}{4\pi t_0}\cdot\frac{\partial}{\partial t_0}Q$$

$$=\frac{1}{t_0}\left(\frac{w}{t_0}+\frac{1}{4\pi t_0}Q\right)-\frac{w}{t_0^2}-\frac{1}{4\pi t_0^2}Q+\frac{1}{4\pi t_0}\cdot\frac{\partial}{\partial t_0}Q=\frac{1}{4\pi t_0}\cdot\frac{\partial}{\partial t_0}Q,$$

and consequently,

$$\frac{\partial^2 w}{\partial t_0^2}=\frac{1}{4\pi t_0}\int_{\partial K(x^0,t_0)}\left(\sum_{k=1}^{3}\frac{\partial^2 f}{\partial x_k^2}(x)\right)ds. \tag{2.5.15}$$

In particular, this yields (as in the proof of (2.5.11)):

$$\left.\frac{\partial^2 w}{\partial t_0^2}\right|_{t_0=0}=0. \tag{2.5.16}$$

Differentiating (2.5.10) twice with respect to x_k^0, we calculate

$$\frac{\partial^2 w}{\partial x_k^{0\,2}}=\frac{t_0}{4\pi}\int_{\partial K(0,1)}\frac{\partial^2 f}{\partial x_k^2}(x^0+t_0\xi)\,ds=\frac{1}{4\pi t_0}\int_{\partial K(x^0,t_0)}\frac{\partial^2 f}{\partial x_k^2}(x)\,ds.$$

Together with (2.5.15) this yields

$$\frac{\partial^2 w}{\partial t_0^2} = \sum_{k=1}^{3} \frac{\partial^2 w}{\partial x_k^{0\,2}}, \tag{2.5.17}$$

i.e. the function w satisfies the wave equation. Moreover, using (2.5.1) we deduce

$$\frac{\partial^2}{\partial t_0^2}\left(\frac{\partial w}{\partial t_0}\right) = \sum_{k=1}^{3} \frac{\partial^2}{\partial x_k^{0\,2}}\left(\frac{\partial w}{\partial t_0}\right), \tag{2.5.18}$$

i.e. the function $\dfrac{\partial w}{\partial t_0}$ also satisfies the wave equation.

Formula (2.5.8) has the form

$$u(x^0,t_0) = \frac{\partial}{\partial t_0} w_1(x^0,t_0) + w_2(x^0,t_0),$$

where the functions w_1 and w_2 have the form (2.5.9) with $f = \varphi$ and $f = \psi$, respectively. Therefore, it follows from (2.5.11), (2.5.13) and (2.5.16)-(2.5.18) that the function u, defined by (2.5.8), satisfies (2.5.1) and (2.5.2). Theorem 2.5.1 is proved. □

Let us study the stability of the solution of the problem (2.5.1)-(2.5.2).

Definition 2.5.2. The solution of problem (2.5.1)-(2.5.2) is called stable if for each $\varepsilon > 0$ and $T > 0$ there exists $\delta = \delta(\varepsilon, T)$ such that if $|\varphi(x) - \tilde{\varphi}(x)| \le \delta$, $|\psi(x) - \tilde{\psi}(x)| \le \delta$, $\left|\frac{\partial\varphi(x)}{\partial x_k} - \frac{\partial\tilde{\varphi}(x)}{\partial x_k}\right| \le \delta$ for all $x \in \mathbf{R}^3$, then $|u(x,t) - \tilde{u}(x,t)| \le \varepsilon$ for all $x \in \mathbf{R}^3$, $0 \le t \le T$. Here $\tilde{u}(x,t)$ is the solution of the Cauchy problem with the initial data $\tilde{\varphi}$ and $\tilde{\psi}$.

Let us show that the solution of problem (2.5.1)-(2.5.2) is stable. Indeed, we rewrite (2.5.8) in the form

$$\begin{aligned} u(x^0,t_0) &= \frac{\partial}{\partial t_0}\left(\frac{t_0}{4\pi}\int_{\partial K(0,1)} \varphi(x^0 + t_0\xi)\,ds\right) + \frac{t_0}{4\pi}\int_{\partial K(0,1)} \psi(x^0 + t_0\xi)\,ds \\ &= \frac{t_0}{4\pi}\int_{\partial K(0,1)} \psi(x^0 + t_0\xi)\,ds + \frac{1}{4\pi}\int_{\partial K(0,1)} \varphi(x^0 + t_0\xi)\,ds \\ &\quad + \frac{t_0}{4\pi}\int_{\partial K(0,1)} \sum_{k=1}^{3} \frac{\partial\varphi}{\partial x_k}(x^0 + t_0\xi)\cdot\xi_k\,ds. \end{aligned}$$

For $\varepsilon > 0, T > 0$ choose $\delta = \varepsilon/(4T+1)$ and suppose that

$$|\varphi(x) - \tilde{\varphi}(x)| \le \delta,\ |\psi(x) - \tilde{\psi}(x)| \le \delta,\ \left|\frac{\partial\varphi(x)}{\partial x_k} - \frac{\partial\tilde{\varphi}(x)}{\partial x_k}\right| \le \delta$$

for all $x \in \mathbf{R}^3$. Then

$$\begin{aligned} |u(x^0,t_0) - \tilde{u}(x^0,t_0)| &\le \frac{t_0}{4\pi}\int_{\partial K(0,1)} |\psi(x^0 + t_0\xi) - \tilde{\psi}(x^0 + t_0\xi)|\,ds \\ &\quad + \frac{1}{4\pi}\int_{\partial K(0,1)} |\varphi(x^0 + t_0\xi) - \tilde{\varphi}(x^0 + t_0\xi)|\,ds \end{aligned}$$

$$+\frac{t_0}{4\pi}\int_{\partial K(0,1)}\left|\sum_{k=1}^{3}\left(\frac{\partial\varphi}{\partial x_k}-\frac{\partial\tilde{\varphi}}{\partial x_k}\right)(x^0+t_0\xi)\cdot\xi_k\right|ds\le\delta(4T+1)=\varepsilon.$$

Thus, the Cauchy problem (2.5.1)-(2.5.2) is well-posed.

Hadamard's method of descent

1) We consider the Cauchy problem for the two-dimensional wave equation:

$$u_{tt}=\Delta u,\quad x\in\mathbf{R}^2,\ t>0, \tag{2.5.19}$$

$$u_{|t=0}=\varphi(x),\quad u_{t|t=0}=\psi(x). \tag{2.5.20}$$

Here $x=(x_1,x_2)\in\mathbf{R}^2$, $t\ge 0$ are independent variables, $u(x_1,x_2,t)$ is an unknown function and $\Delta u:=\dfrac{\partial^2 u}{\partial x_1^2}+\dfrac{\partial^2 u}{\partial x_2^2}$ is the two-dimensional Laplace operator. The solution of problem (2.5.19)-(2.5.20) can be obtained by a method which is similar to the one used above. However, it is more convenient to obtain the solution of problem (2.5.19)-(2.5.20) directly from Kirchhoff's formula (2.5.8) (i.e. consider problem (2.5.19)-(2.5.20) as a particular case of problem (2.5.1)-(2.5.2)).

For this purpose we consider problem (2.5.1)-(2.5.2) and suppose that the functions φ and ψ do not depend on x_3. Let us show that in this case the function $u(x^0,t_0)$, defined by (2.5.8), does not depend on x_3^0, i.e. it is a solution of problem (2.5.19)-(2.5.20). Denote $\rho=\sqrt{(x_1-x_1^0)^2+(x_1-x_1^0)^2}$, $\sigma=\sigma(x_1^0,x_2^0,t_0)=\{(x_1,x_2):\ \rho\le t_0\}$ is the disc of radius t_0 with center (x_1^0,x_2^0). Since $\partial K(x^0,t_0)=S^+\cup S^-$, where $S^\pm$ are the half-spheres $x_3=x_3^0\pm\sqrt{t_0^2-\rho^2}$ (see fig. 2.5.2), it follows from (2.5.9) that

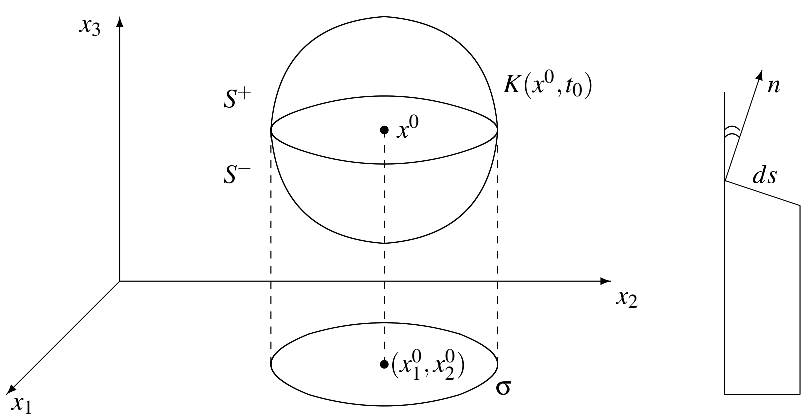

Figure 2.5.2.

$$w(x^0,t_0)=\frac{1}{4\pi}\int_{S^+}\frac{f(x)}{t_0}ds+\frac{1}{4\pi}\int_{S^-}\frac{f(x)}{t_0}ds$$
$$=\frac{1}{4\pi t_0}\int_{\sigma}f(x_1,x_2,x_3^0+\sqrt{t_0^2-\rho^2})\frac{dx_1dx_2}{\cos(n,x_3)}$$

$$+\frac{1}{4\pi t_0}\int_\sigma f(x_1,x_2,x_3^0-\sqrt{t_0^2-\rho^2})\frac{dx_1dx_2}{\cos(n,x_3)}.$$

Let f be independent of x_3. Since $\cos(\bar{n},\bar{x}_3)=\dfrac{\sqrt{t_0^2-\rho^2}}{t_0}$ (see fig. 2.5.3), we have

$$w(x^0,t_0)=\frac{1}{2\pi}\int_{\sigma(x_1^0,x_2^0,t_0)}\frac{f(x_1,x_2)}{\sqrt{t_0^2-\rho^2}}\,dx_1dx_2.$$

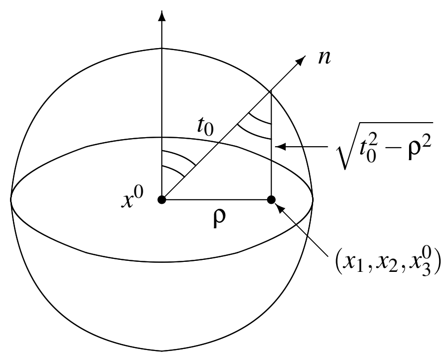

Figure 2.5.3.

In particular, this yields that the function $w(x^0,t_0)$ does not depend on x_3^0. Thus, if in (2.5.1)-(2.5.2) the initial data φ and ψ do not depend on x_3, then the solution also does not depend on x_3, and Kirchhoff's formula (2.5.8) takes the form

$$u(x_1^0,x_2^0,t_0)=\frac{1}{2\pi}\frac{\partial}{\partial t_0}\int_{\sigma(x_1^0,x_2^0,t_0)}\frac{\varphi(x_1,x_2)}{\sqrt{t_0^2-\rho^2}}\,dx_1dx_2$$

$$+\frac{1}{2\pi}\int_{\sigma(x_1^0,x_2^0,t_0)}\frac{\psi(x_1,x_2)}{\sqrt{t_0^2-\rho^2}}\,dx_1dx_2. \qquad (2.5.21)$$

Formula (2.5.21) is called the *Poisson formula*. It represents the solution of the Cauchy problem (2.5.19)-(2.5.20).

2) Let us make one more step in Hadamard's method of descent. Suppose that in (2.5.21) the functions φ and ψ do not depend on x_2. Then

$$J:=\frac{1}{2\pi}\int_{\sigma(x_1^0,x_2^0,t_0)}\frac{\varphi(x_1)}{\sqrt{t_0^2-\rho^2}}\,dx_1dx_2$$

$$=\frac{1}{2\pi}\int_{x_1^0-t_0}^{x_1^0+t_0}\varphi(x_1)\,dx_1\int_{x_2^0-h}^{x_2^0+h}\frac{dx_2}{\sqrt{t_0^2-\rho^2}},$$

where $h^2=t_0^2-(x_1-x_1^0)^2$ (see fig. 2.5.4).

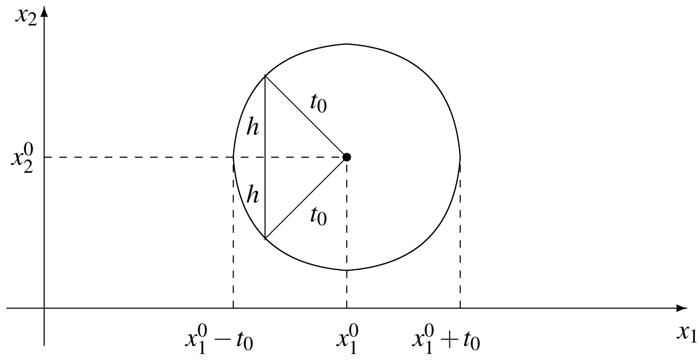

Figure 2.5.4.

Since

$$\int_{x_2^0-h}^{x_2^0+h} \frac{dx_2}{\sqrt{t_0^2-\rho^2}} = \arcsin \frac{x_2 - x_2^0}{h}\bigg|_{x_2^0-h}^{x_2^0+h} = \frac{\pi}{2} - \Big(-\frac{\pi}{2}\Big) = \pi,$$

we have

$$J = \frac{1}{2}\int_{x_1^0-t_0}^{x_1^0+t_0} \varphi(x_1)\,dx_1.$$

Analogously one transforms the second integral in (2.5.21). Thus, (2.5.21) takes the form

$$u(x_1^0,t_0) = \frac{1}{2}\Big(\varphi(x_1^0+t_0)+\varphi(x_1^0-t_0)\Big) + \frac{1}{2}\int_{x_1^0-t_0}^{x_1^0+t_0} \psi(x_1)\,dx_1. \tag{2.5.22}$$

Formula (2.5.22) is the D'Alembert formula (see Section 2.1), which gives the solution of the Cauchy problem for the equation of a vibrating string:

$$\left.\begin{aligned} u_{tt} &= u_{xx}, \quad -\infty < x < \infty,\ t > 0,\\ u_{|t=0} &= \varphi(x), \quad u_{t|t=0} = \psi(x). \end{aligned}\right\} \tag{2.5.23}$$

Remark 2.5.1. The wave equations for $n=3, n=2$ and $n=1$ are called the equations of spherical, cylindrical and plane waves, respectively. The formulae (2.5.8), (2.5.21) and (2.5.22) give us an opportunity to study the physical picture of wave propagation. We note that for $n=3,$ the solution $u(x^0,t_0)$ depends on the initial data given only on the boundary of the characteristic cone (i.e. on the sphere $\partial K(x^0,t_0)$). For $n=2$ and $n=1,$ the solution $u(x^0,t_0)$ depends on the initial data given on the whole base of the characteristic cone (i.e. the circle $\sigma(x_1^0,x_2^0,t_0)$ or the segment $[x_1^0-t_0,x_1^0+t_0]$). In other words, for $n=3$ initial perturbations localized in the space produce in each point x^0 perturbations localized with respect to time (*Huygens's principle*), i.e. the wave has front and back wave fronts (leading edge and trailing edge). For $n=2,$ initial perturbations localized in the space are not localized with respect to time (the Huygens principle is not valid), i.e. the wave has a leading edge and has no trailing edge - the oscillations will continue for infinitely long time. We note that the problem for $n=2$ can be considered as a three-dimensional problem with initial data given in an infinite cylinder which do not depend on the third coordinate.

2.6. An Inverse Problem for the Wave Equation

Sections 2.6-2.9 contain a material for advanced studies, and they can be omitted "in the first reading". These sections are devoted to studying inverse problems for differential equations. The Inverse problems that we study below consist in recovering coefficients of differential equations from characteristics which can be measured. Such problems often appear in various areas of natural sciences and engineering (see [15]-[22] and the references therein).

In this section we consider an inverse problem for a wave equation with a focused source of disturbance. In applied problems the data are often functions of compact support localized within a relative small area of space. It is convenient to model such situations mathematically as problems with a focused source of disturbance.

Consider the following boundary value problem $B(q(x),h)$:

$$u_{tt} = u_{xx} - q(x)u, \quad 0 \le x \le t, \tag{2.6.1}$$

$$u(x,x) = 1, \quad (u_x - hu)_{|x=0,} \tag{2.6.2}$$

where $q(x)$ is a complex-valued locally integrable function (i.e. it is integrable on every finite interval), and h is a complex number. Denote $r(t) := u(0,t)$. The function r is called the trace of the solution. In this section we study the following inverse problem.

Inverse Problem 2.6.1. Given the trace $r(t)$, $t \ge 0$, of the solution of $B(q(x),h)$, construct $q(x)$, $x \ge 0$, and h.

We prove an uniqueness theorem for Inverse Problem 2.6.1 (Theorem 2.6.3), provide an algorithm for the solution of this inverse problem (Algorithm 2.6.1) and give necessary and sufficient conditions for its solvability (Theorem 2.6.4).

Remark 2.6.1. Let us note here that the boundary value problem $B(q(x),h)$ is equivalent to a Cauchy problem with a focused source of disturbance. For simplicity, we assume here that $h = 0$. We define $u(x,t) = 0$ for $0 < t < x$, and $u(x,t) = u(-x,t)$, $q(x) = q(-x)$ for $x < 0$. Then, using symmetry, it follows that $u(x,t)$ is a solution of the Goursat problem

$$u_{tt} = u_{xx} - q(x)u, \quad 0 \le |x| \le t,$$

$$u(x,|x|) = 1.$$

Moreover, it can be shown that this Goursat problem is equivalent to the Cauchy problem

$$u_{tt} = u_{xx} - q(x)u, \quad -\infty < x < \infty,\ t > 0,$$

$$u_{|t=0} = 0, \quad u_{t|t=0} = 2\delta(x),$$

where $\delta(x)$ is the Dirac delta-function. Similarly, for $h \neq 0$, the boundary value problem (2.6.1)-(2.6.2) also corresponds to a problem with a focused source of disturbance.

Let us return to the boundary value problem (2.6.1)-(2.6.2). Denote

$$Q(x) = \int_0^x |q(t)|\,dt, \quad Q_*(x) = \int_0^x Q(t)\,dt, \quad d = \max(0,-h).$$

Theorem 2.6.1. *The boundary value problem* $(2.6.1)-(2.6.2)$ *has a unique solution* $u(x,t)$, *and*

$$|u(x,t)| \leq \exp(d(t-x))\exp\left(2Q_*\left(\frac{t+x}{2}\right)\right), \quad 0 \leq x \leq t. \tag{2.6.3}$$

Proof. We transform (2.6.1)-(2.6.2) by means of the replacement

$$\xi = t+x,\ \eta = t-x, \quad v(\xi,\eta) = u\left(\frac{\xi-\eta}{2}, \frac{\xi+\eta}{2}\right)$$

to the boundary value problem

$$v_{\xi\eta}(\xi,\eta) = -\frac{1}{4}q\left(\frac{\xi-\eta}{2}\right)v(\xi,\eta), \quad 0 \leq \eta \leq \xi, \tag{2.6.4}$$

$$v(\xi,0) = 1, \quad (v_\xi(\xi,\eta) - v_\eta(\xi,\eta) - hv(\xi,\eta))_{|\xi=\eta} = 0. \tag{2.6.5}$$

Since $v_\xi(\xi,0) = 0$, integration of (2.6.4) with respect to η gives

$$v_\xi(\xi,\eta) = -\frac{1}{4}\int_0^\eta q\left(\frac{\xi-\alpha}{2}\right)v(\xi,\alpha)\,d\alpha. \tag{2.6.6}$$

In particular, we have

$$v_\xi(\xi,\eta)_{|\xi=\eta} = -\frac{1}{4}\int_0^\eta q\left(\frac{\eta-\alpha}{2}\right)v(\eta,\alpha)\,d\alpha. \tag{2.6.7}$$

It follows from (2.6.6) that

$$v(\xi,\eta) = v(\eta,\eta) - \frac{1}{4}\int_\eta^\xi\left(\int_0^\eta q\left(\frac{\beta-\alpha}{2}\right)v(\beta,\alpha)\,d\alpha\right)d\beta. \tag{2.6.8}$$

Let us calculate $v(\eta,\eta)$. Since

$$\frac{d}{d\eta}(v(\eta,\eta)\exp(h\eta)) = (v_\xi(\xi,\eta) + v_\eta(\xi,\eta) + hv(\xi,\eta))_{|\xi=\eta}\exp(h\eta),$$

we get by virtue of (2.6.5) and (2.6.7),

$$\frac{d}{d\eta}(v(\eta,\eta)\exp(h\eta)) = 2v_\xi(\xi,\eta)_{|\xi=\eta}\exp(h\eta)$$

$$= -\frac{1}{2}\exp(h\eta)\int_0^\eta q\left(\frac{\eta-\alpha}{2}\right)v(\eta,\alpha)\,d\alpha.$$

This yields (with $v(0,0) = 1$)

$$v(\eta,\eta)\exp(h\eta) - 1 = -\frac{1}{2}\int_0^\eta \exp(h\beta)\left(\int_0^\beta q\left(\frac{\beta-\alpha}{2}\right)v(\beta,\alpha)\,d\alpha\right)d\beta,$$

and consequently

$$v(\eta,\eta)=\exp(-h\eta)$$
$$-\frac{1}{2}\int_0^{\eta}\exp(-h(\eta-\beta))\left(\int_0^{\beta}q\left(\frac{\beta-\alpha}{2}\right)v(\beta,\alpha)\,d\alpha\right)d\beta. \tag{2.6.9}$$

Substituting (2.6.9) into (2.6.8) we deduce that the function $v(\xi,\eta)$ satisfies the integral equation

$$v(\xi,\eta)=\exp(-h\eta)$$
$$-\frac{1}{2}\int_0^{\eta}\exp(-h(\eta-\beta))\left(\int_0^{\beta}q\left(\frac{\beta-\alpha}{2}\right)v(\beta,\alpha)\,d\alpha\right)d\beta$$
$$-\frac{1}{4}\int_{\eta}^{\xi}\left(\int_0^{\eta}q\left(\frac{\beta-\alpha}{2}\right)v(\beta,\alpha)\,d\alpha\right)d\beta. \tag{2.6.10}$$

Conversely, if $v(\xi,\eta)$ is a solution of (2.6.10) then one can verify that $v(\xi,\eta)$ satisfies (2.6.4)-(2.6.5).

We solve the integral equation (2.6.10) by the method of successive approximations. The calculations are slightly different for $h\ge 0$ and $h<0$.

Case 1. Let $h\ge 0$. Denote

$$v_0(\xi,\eta)=\exp(-h\eta),$$

$$v_{k+1}(\xi,\eta)=-\frac{1}{2}\int_0^{\eta}\exp(-h(\eta-\beta))\left(\int_0^{\beta}q\left(\frac{\beta-\alpha}{2}\right)v_k(\beta,\alpha)\,d\alpha\right)d\beta$$
$$-\frac{1}{4}\int_{\eta}^{\xi}\left(\int_0^{\eta}q\left(\frac{\beta-\alpha}{2}\right)v_k(\beta,\alpha)\,d\alpha\right)d\beta. \tag{2.6.11}$$

Let us show by induction that

$$|v_k(\xi,\eta)|\le\frac{1}{k!}\left(2Q_*\left(\frac{\xi}{2}\right)\right)^k,\quad k\ge 0,\ 0\le\eta\le\xi. \tag{2.6.12}$$

Indeed, for $k=0$, (2.6.12) is obvious. Suppose that (2.6.12) is valid for a certain $k\ge 0$. It follows from (2.6.11) that

$$|v_{k+1}(\xi,\eta)|\le\frac{1}{2}\int_0^{\xi}\left(\int_0^{\eta}\left|q\left(\frac{\beta-\alpha}{2}\right)v_k(\beta,\alpha)\right|d\alpha\right)d\beta. \tag{2.6.13}$$

Substituting (2.6.12) into the right-hand side of (2.6.13) we obtain

$$|v_{k+1}(\xi,\eta)|\le\frac{1}{2k!}\int_0^{\xi}\left(2Q_*\left(\frac{\beta}{2}\right)\right)^k\left(\int_0^{\eta}\left|q\left(\frac{\beta-\alpha}{2}\right)\right|d\alpha\right)d\beta$$
$$\le\frac{1}{k!}\int_0^{\xi}\left(2Q_*\left(\frac{\beta}{2}\right)\right)^k\left(\int_0^{\beta/2}|q(s)|\,ds\right)d\beta$$
$$=\frac{1}{k!}\int_0^{\xi}\left(2Q_*\left(\frac{\beta}{2}\right)\right)^k Q\left(\frac{\beta}{2}\right)d\beta$$

$$= \frac{1}{k!}\int_0^{\xi/2}(2Q_*(s))^k(2Q_*(s))'\,ds = \frac{1}{(k+1)!}\left(2Q_*\left(\frac{\xi}{2}\right)\right)^{k+1};$$

hence (2.6.12) is valid.

It follows from (2.6.12) that the series

$$v(\xi,\eta) = \sum_{k=0}^{\infty} v_k(\xi,\eta)$$

converges absolutely and uniformly on compact sets $0 \le \eta \le \xi \le T$, and

$$|v(\xi,\eta)| \le \exp\left(2Q_*\left(\frac{\xi}{2}\right)\right).$$

The function $v(\xi,\eta)$ is the unique solution of the integral equation (2.6.10). Consequently, the function $u(x,t) = v(t+x,t-x)$ is the unique solution of the boundary value problem (2.6.1)-(2.6.2), and (2.6.3) holds.

Case 2. Let $h < 0$. we transform (2.6.10) by means of the replacement

$$w(\xi,\eta) = v(\xi,\eta)\exp(h\eta)$$

to the integral equation

$$w(\xi,\eta) = 1 - \frac{1}{2}\int_0^{\eta}\left(\int_0^{\beta} q\left(\frac{\beta-\alpha}{2}\right)\exp(h(\beta-\alpha))w(\beta,\alpha)\,d\alpha\right)d\beta$$

$$-\frac{1}{4}\int_{\eta}^{\xi}\left(\int_0^{\eta} q\left(\frac{\beta-\alpha}{2}\right)\exp(h(\eta-\alpha))w(\beta,\alpha)\,d\alpha\right)d\beta. \qquad (2.6.14)$$

By the method of successive approximations we get similarly to Case 1 that the integral equation (2.6.14) has a unique solution, and that

$$|w(\xi,\eta)| \le \exp\left(2Q_*\left(\frac{\xi}{2}\right)\right),$$

i.e. Theorem 2.6.1 is proved also for $h < 0$. □

Remark 2.6.2. It follows from the proof of Theorem 2.6.1 that the solution $u(x,t)$ of (2.6.1)-(2.6.2) in the domain $\Theta_T := \{(x,t):\ 0 \le x \le t,\ 0 \le x+t \le 2T\}$ is uniquely determined by the specification of h and $q(x)$ for $0 \le x \le T$, i.e. if $q(x) = \tilde{q}(x),\ x \in [0,T]$ and $h = \tilde{h}$, then $u(x,t) = \tilde{u}(x,t)$ for $(x,t) \in \Theta_T$. Therefore, one can also consider the boundary value problem (2.6.1)-(2.6.2) in the domains Θ_T and study the inverse problem of recovering $q(x),\ 0 \le x \le T$ and h from the given trace $r(t),\ t \in [0,2T]$.

Denote by $D_N\ (N \ge 0)$ the set of functions $f(x),\ x \ge 0$ such that for each fixed $T > 0$, the functions $f^{(j)}(x),\ j = \overline{0,N-1}$ are absolutely continuous on $[0,T]$, i.e. $f^{(j)}(x) \in L(0,T),\ j = \overline{0,N}$. It follows from the proof of Theorem 2.6.1 that $r(t) \in D_2,\ r(0) = 1,\ r'(0) = -h$. Moreover, the function r'' has the same smoothness properties as the potential q. For example, if $q \in D_N$ then $r \in D_{N+2}$.

In order to solve Inverse Problem 2.6.1 we will use the Riemann formula for the solution of the Cauchy problem

$$\left.\begin{array}{c} u_{tt}-p(t)u=u_{xx}-q(x)u+p_1(x,t),\\ -\infty<x<\infty,\ t>0,\\ u_{|t=0}=r(x),\quad u_{t|t=0}=s(x). \end{array}\right\} \tag{2.6.15}$$

According to the Riemann formula (see Section 2.4) the solution of (2.6.15) has the form

$$u(x,t)=\frac{1}{2}\big(r(x+t)+r(x-t)\big)$$

$$+\frac{1}{2}\int_{x-t}^{x+t}\big(s(\xi)R(\xi,0,x,t)-r(\xi)R_2(\xi,0,x,t)\big)\,d\xi$$

$$+\frac{1}{2}\int_0^t d\tau\int_{x+\tau-t}^{x+t-\tau}R(\xi,\tau,x,t)p_1(\xi,\tau)\,d\xi,$$

where $R(\xi,\tau,x,t)$ is the Riemann function, and $R_2=\dfrac{\partial R}{\partial\tau}$. Note that if $q(x)\equiv const$, then $R(\xi,\tau,x,t)=R(\xi-x,\tau,t)$. In particular, the solution of the Cauchy problem

$$\left.\begin{array}{c} u_{tt}=u_{xx}-q(x)u,\quad -\infty<t<\infty,\ x>0,\\ u_{|x=0}=r(t),\quad u_{x|x=0}=hr(t), \end{array}\right\} \tag{2.6.16}$$

has the form

$$u(x,t)=\frac{1}{2}\big(r(t+x)+r(t-x)\big)$$

$$-\frac{1}{2}\int_{t-x}^{t+x}r(\xi)\big(R_2(\xi-t,0,x)-hR(\xi-t,0,x)\big)\,d\xi.$$

The change of variables $\tau=t-\xi$ leads to

$$u(x,t)=\frac{1}{2}\big(r(t+x)+r(t-x)\big)+\frac{1}{2}\int_{-x}^{x}r(t-\tau)G(x,\tau)\,d\tau, \tag{2.6.17}$$

where $G(x,\tau)=-R_2(-\tau,0,x)+hR(-\tau,0,x)$.

Let us take $r(t)=\cos\rho t$ in (2.6.16), and let $\varphi(x,\lambda)$ be the solution of the equation

$$-\varphi''(x,\lambda)+q(x)\varphi(x,\lambda)=\lambda\varphi(x,\lambda),\quad \lambda=\rho^2,$$

under the initial conditions $\varphi(0,\lambda)=1$, $\varphi'(0,\lambda)=h$. Obviously, the function $u(x,t)=\varphi(x,\lambda)\cos\rho t$ is a solution of problem (2.6.16). Therefore, (2.6.17) yields for $t=0$,

$$\varphi(x,\lambda)=\cos\rho x+\frac{1}{2}\int_{-x}^{x}G(x,\tau)\cos\rho\tau\,d\tau.$$

Since $G(x,-\tau)=G(x,\tau)$, we have

$$\varphi(x,\lambda)=\cos\rho x+\int_0^x G(x,\tau)\cos\rho\tau\,d\tau.$$

Hence

$$\varphi'(x,\lambda) = -\rho\sin\rho x + G(x,x)\cos\rho x + \int_0^x G_x(x,\tau)\cos\rho\tau\,d\tau,$$

$$G(0,0) = h,$$

$$\varphi''(x,\lambda) = -\rho^2\cos\rho x - G(x,x)\rho\sin\rho x + \frac{dG(x,x)}{dx}\cos\rho x$$

$$+G_x(x,t)_{|t=x}\cos\rho x + \int_0^x G_{xx}(x,\tau)\cos\rho\tau\,d\tau,$$

$$\lambda\varphi(x,\lambda) = \rho^2\cos\rho x + \rho^2\int_0^x G(x,\tau)\cos\rho\tau\,d\tau.$$

Integration by parts yields

$$\lambda\varphi(x,\lambda) = \rho^2\cos\rho x + G(x,x)\rho\sin\rho x + G_t(x,t)_{|t=x}\cos\rho x$$

$$-G_t(x,t)_{|t=0} - \int_0^x G_{tt}(x,\tau)\cos\rho\tau\,d\tau.$$

Since $\varphi''(x,\lambda) + \lambda\varphi(x,\lambda) - q(x)\varphi(x,\lambda) = 0$ and

$$(G_t(x,t) + G_x(x,t))_{|t=x} = \frac{dG(x,x)}{dx}$$

it follows that

$$\Big(2\frac{dG(x,x)}{dx} - q(x)\Big)\cos\rho x - G_t(x,t)_{|t=0}$$

$$+\int_0^x \Big(G_{xx}(x,\tau) - G_{tt}(x,\tau) - q(x)G(x,\tau)\Big)\cos\rho\tau\,d\tau = 0,$$

and consequently,

$$2\frac{dG(x,x)}{dx} = q(x), \quad G_t(x,t)_{|t=0} = 0,$$

$$G_{tt}(x,\tau) = G_{xx}(x,\tau) - q(x)G(x,\tau).$$

Thus, we have proved that

$$q(x) = 2\frac{dG(x,x)}{dx}, \quad G(0,0) = h,$$

hence

$$G(x,x) = h + \frac{1}{2}\int_0^x q(t)\,dt. \tag{2.6.18}$$

Let us go on to the solution of Inverse Problem 2.6.1. Let $u(x,t)$ be the solution of the boundary value problem (2.6.1)-(2.6.2). We define $u(x,t) = 0$ for $0 \le t < x$, and $u(x,t) = -u(x,-t)$, $r(t) = -r(-t)$ for $t < 0$. Then $u(x,t)$ is the solution of the Cauchy problem (2.6.16), and consequently (2.6.17) holds. But $u(x,t) = 0$ for $x > |t|$ (this is a connection between $q(x)$ and $r(t)$), and hence

$$\frac{1}{2}\left(r(t+x) + r(t-x)\right) + \frac{1}{2}\int_{-x}^x r(t-\tau)G(x,\tau)\,d\tau = 0,\ |t| < x. \tag{2.6.19}$$

Denote $a(t) = r'(t)$. Differentiating (2.6.19) with respect to t and using the relations

$$r(0+) = 1, \quad r(0-) = -1, \tag{2.6.20}$$

we obtain

$$G(x,t) + F(x,t) + \int_0^x G(x,\tau)F(t,\tau)\,d\tau = 0, \quad 0 < t < x, \tag{2.6.21}$$

where

$$F(x,t) = \frac{1}{2}\left(a(t+x) + a(t-x)\right) \tag{2.6.22}$$

with $a(t) = r'(t)$. Equation (2.6.21) is called the *Gelfand-Levitan equation* for the Inverse Problem 2.6.1.

Theorem 2.6.2. *For each fixed* $x > 0$, *equation* (2.6.21) *has a unique solution.*

Proof. Fix $x_0 > 0$. It is sufficient to prove that the homogeneous equation

$$g(t) + \int_0^{x_0} g(\tau)F(t,\tau)\,d\tau = 0, \quad 0 \le t \le x_0 \tag{2.6.23}$$

has only the trivial solution $g(t) = 0$.

Let $g(t)$, $0 \le t \le x_0$ be a solution of (2.6.23). Since $a(t) = r'(t) \in D_1$, it follows from (2.6.22) and (2.6.23) that $g(t)$ is an absolutely continuous function on $[0, x_0]$. We define $g(-t) = g(t)$ for $t \in [0, x_0]$, and furthermore $g(t) = 0$ for $|t| > x_0$.

Let us show that

$$\int_{-x_0}^{x_0} r(t-\tau)g(\tau)\,d\tau = 0, \quad t \in [-x_0, x_0]. \tag{2.6.24}$$

Indeed, by virtue of (2.6.20) and (2.6.23), we have

$$\frac{d}{dt}\left(\int_{-x_0}^{x_0} r(t-\tau)g(\tau)\,d\tau\right)$$

$$= \frac{d}{dt}\left(\int_{-x_0}^{t} r(t-\tau)g(\tau)\,d\tau + \int_{t}^{x_0} r(t-\tau)g(\tau)\,d\tau\right)$$

$$= r(+0)g(t) - r(-0)g(t) + \int_{-x_0}^{x_0} a(t-\tau)g(\tau)\,d\tau$$

$$= 2\left(g(t) + \int_0^{x_0} g(\tau)F(t,\tau)\,d\tau\right) = 0.$$

Consequently,

$$\int_{-x_0}^{x_0} r(t-\tau)g(\tau)\,d\tau \equiv C_0.$$

Taking here $t = 0$ and using that $r(-\tau)g(\tau)$ is an odd function, we calculate $C_0 = 0$, i.e. (2.6.24) is valid.

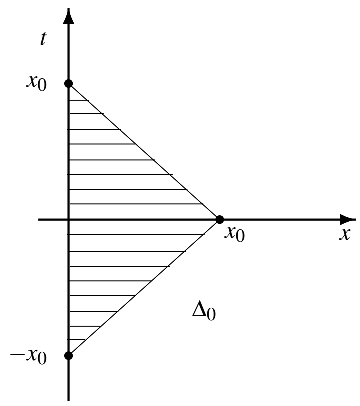

Figure 2.6.1.

Denote $\Delta_0 = \{(x,t) : x - x_0 \le t \le x_0 - x,\ 0 \le x \le x_0\}$ and consider the function

$$w(x,t) := \int_{-\infty}^{\infty} u(x,t-\tau)g(\tau)\,d\tau, \quad (x,t) \in \Delta_0, \tag{2.6.25}$$

where $u(x,t)$ is the solution of the boundary value problem (2.6.1)-(2.6.2). Let us show that

$$w(x,t) = 0, \quad (x,t) \in \Delta_0. \tag{2.6.26}$$

Since $u(x,t) = 0$ for $x > |t|$, (2.6.25) takes the form

$$w(x,t) = \int_{-\infty}^{t-x} u(x,t-\tau)g(\tau)\,d\tau + \int_{t+x}^{\infty} u(x,t-\tau)g(\tau)\,d\tau. \tag{2.6.27}$$

Differentiating (2.6.27) and using the relations

$$u(x,x) = 1, \quad u(x,-x) = -1, \tag{2.6.28}$$

we calculate

$$w_x(x,t) = g(t+x) - g(t-x)$$
$$+ \int_{-\infty}^{t-x} u_x(x,t-\tau)g(\tau)\,d\tau + \int_{t+x}^{\infty} u_x(x,t-\tau)g(\tau)\,d\tau, \tag{2.6.29}$$
$$w_t(x,t) = g(t+x) + g(t-x)$$
$$+ \int_{-\infty}^{t-x} u_t(x,t-\tau)g(\tau)\,d\tau + \int_{t+x}^{\infty} u_t(x,t-\tau)g(\tau)\,d\tau. \tag{2.6.30}$$

Since, in view of (2.6.28),

$$\Big(u_x(x,t) \pm u_t(x,t)\Big)_{|t=\pm x} = \frac{d}{dx}u(x,\pm x) = 0,$$

it follows from (2.6.29)-(2.6.30) that

$$w_{xx}(x,t) - w_{tt}(x,t) - q(x)w(x,t)$$

$$= \int_{-\infty}^{\infty} [u_{xx} - u_{tt} - q(x)u](x, t-\tau) g(\tau)\, d\tau,$$

and consequently

$$w_{tt}(x,t) = w_{xx}(x,t) - q(x)w(x,t), \quad (x,t) \in \Delta_0. \tag{2.6.31}$$

Furthermore, (2.6.25) and (2.6.29) yield

$$w(0,t) = \int_{-x_0}^{x_0} r(t-\tau) g(\tau)\, d\tau,$$

$$w_x(0,t) = h \int_{-x_0}^{x_0} r(t-\tau) g(\tau)\, d\tau = 0, \quad t \in [-x_0, x_0].$$

Therefore, according to (2.6.24), we have

$$w(0,t) = w_x(0,t) = 0, \quad t \in [-x_0, x_0]. \tag{2.6.32}$$

Since the Cauchy problem (2.6.31)-(2.6.32) has only the trivial solution, we arrive at (2.6.26).

Denote $u_1(x,t) := u_t(x,t)$. It follows from (2.6.30) that

$$w_t(x,0) = 2g(x) + \int_{-\infty}^{-x} u_1(x,\tau) g(\tau)\, d\tau + \int_{x}^{\infty} u_1(x,\tau) g(\tau)\, d\tau$$

$$= 2(g(x) + \int_{x}^{\infty} u_1(x,\tau) g(\tau)\, d\tau) = 2(g(x) + \int_{x}^{x_0} u_1(x,\tau) g(\tau)\, d\tau).$$

Taking (2.6.26) into account we get

$$g(x) + \int_{x}^{x_0} u_1(x,\tau) g(\tau)\, d\tau = 0, \quad 0 \le x \le x_0.$$

This integral equation has only the trivial solution $g(x) = 0$, and consequently Theorem 2.6.2 is proved. □

Let r and $\tilde{r}$ be the traces for the boundary value problems $B(q(x),h)$ and $B(\tilde{q}(x),\tilde{h})$ respectively. We agree that if a certain symbol α denotes an object related to $B(q(x),h)$, then $\tilde{\alpha}$ will denote the analogous object related to $B(\tilde{q}(x),\tilde{h})$.

Theorem 2.6.3. *If $r(t) = \tilde{r}(t)$, $t \ge 0$, then $q(x) = \tilde{q}(x)$, $x \ge 0$ and $h = \tilde{h}$. Thus, the specification of the trace r uniquely determines the potential q and the coefficient h.*

Proof. Since $r(t) = \tilde{r}(t)$, $t \ge 0$, we have, by virtue of (2.6.22),

$$F(x,t) = \tilde{F}(x,t).$$

Therefore, Theorem 2.6.2 gives us

$$G(x,t) = \tilde{G}(x,t), \quad 0 \le t \le x. \tag{2.6.33}$$

By virtue of (2.6.18),

$$q(x) = 2\frac{d}{dx}G(x,x), \quad h = G(0,0) = -r'(0). \tag{2.6.34}$$

Together with (2.6.33) this yields $q(x) = \tilde{q}(x),\ x \geq 0$ and $h = \tilde{h}$. □

The Gelfand-Levitan equation (2.6.21) and Theorem 2.6.2 yield finally the following algorithm for the solution of Inverse Problem 2.6.1.

Algorithm 2.6.1. Let $r(t),\ t \geq 0$ be given. Then
(1) Construct the function $F(x,t)$ using (2.6.22).
(2) Find the function $G(x,t)$ by solving equation (2.6.21).
(3) Calculate $q(x)$ and h by the formulae (2.6.34).

Let us now formulate necessary and sufficient conditions for the solvability of Inverse Problem 2.6.1.

Theorem 2.6.4. *For a function $r(t),\ t \geq 0$ to be the trace for a certain boundary value problem $B(q(x),h)$ of the form $(2.6.1)-(2.6.2)$ with $q \in D_N$, it is necessary and sufficient that $r(t) \in D_{N+2},\ r(0) = 1,$ and that for each fixed $x > 0$ the integral equation* (2.6.21) *is uniquely solvable.*

Proof. The necessity part of Theorem 2.6.4 was proved above, here we prove the sufficiency. For simplicity let $N \geq 1$ (the case $N = 0$ requires small modifications). Let a function $r(t),\ t \geq 0,$ satisfying the hypothesis of Theorem 2.6.4, be given, and let $G(x,t),\ 0 \leq t \leq x,$ be the solution of (2.6.21). We define $G(x,t) = G(x,-t),\ r(t) = -r(-t)$ for $t < 0,$ and consider the function

$$u(x,t) := \frac{1}{2}\left(r(t+x) + r(t-x)\right) + \frac{1}{2}\int_{-x}^{x} r(t-\tau)G(x,\tau)\,d\tau, \tag{2.6.35}$$

$$-\infty < t < \infty,\ x \geq 0.$$

Furthermore, we construct q and h via (2.6.34) and consider the boundary value problem (2.6.1)-(2.6.2) with these q and h. Let $\tilde{u}(x,t)$ be the solution of (2.6.1)-(2.6.2), and let $\tilde{r}(t) := \tilde{u}(0,t)$. Our goal is to prove that $\tilde{u} = u,\ \tilde{r} = r.$

Differentiating (2.6.35) and taking (2.6.20) into account, we get

$$u_t(x,t) = \frac{1}{2}\left(a(t+x) + a(t-x)\right) + G(x,t)$$
$$+\frac{1}{2}\int_{-x}^{x} a(t-\tau)G(x,\tau)\,d\tau, \tag{2.6.36}$$

$$u_x(x,t) = \frac{1}{2}\left(a(t+x) - a(t-x)\right)$$
$$+\frac{1}{2}\left(r(t-x)G(x,x) + r(t+x)G(x,-x)\right)$$
$$+\frac{1}{2}\int_{-x}^{x} r(t-\tau)G_x(x,\tau)\,d\tau. \tag{2.6.37}$$

Since $a(0+) = a(0-)$, it follows from (2.6.36) that

$$u_{tt}(x,t) = \frac{1}{2}\left(a'(t+x) + a'(t-x)\right) + G_t(x,t)$$
$$+\frac{1}{2}\int_{-x}^{x} a'(t-\tau)G(x,\tau)\,d\tau. \tag{2.6.38}$$

Integration by parts yields

$$u_{tt}(x,t) = \frac{1}{2}(a'(t+x) + a'(t-x)) + G_t(x,t)$$
$$-\frac{1}{2}(a(t-\tau)G(x,\tau)|_{-x}^{t} + a(t-\tau)G(x,\tau)|_{t}^{x})$$
$$+\frac{1}{2}\int_{-x}^{x} a(t-\tau)G_t(x,\tau)\,d\tau = \frac{1}{2}(a'(t+x) + a'(t-x)) + G_t(x,t)$$
$$+\frac{1}{2}(a(t+x)G(x,-x) - a(t-x)G(x,x))$$
$$+\frac{1}{2}\int_{-x}^{x} r'(t-\tau)G_t(x,\tau)\,d\tau.$$

Integrating by parts again and using (2.6.20) we calculate

$$u_{tt}(x,t) = \frac{1}{2}\Big(a'(t+x) + a'(t-x)\Big) + G_t(x,t)$$
$$+\frac{1}{2}\Big(a(t+x)G(x,-x) - a(t-x)G(x,x)\Big)$$
$$-\frac{1}{2}\Big(r(t-\tau)G_t(x,\tau)\Big|_{-x}^{t} + r(t-\tau)G_t(x,\tau)\Big|_{t}^{x}\Big)$$
$$+\frac{1}{2}\int_{-x}^{x} r(t-\tau)G_{tt}(x,\tau)\,d\tau = \frac{1}{2}\Big(a'(t+x) + a'(t-x)\Big)$$
$$+\frac{1}{2}\Big(a(t+x)G(x,-x) - a(t-x)G(x,x)\Big)$$
$$+\frac{1}{2}\Big(r(t+x)G_t(x,-x) - r(t-x)G_t(x,x)\Big)$$
$$+\frac{1}{2}\int_{-x}^{x} r(t-\tau)G_{tt}(x,\tau)\,d\tau. \tag{2.6.39}$$

Differentiating (2.6.37) we obtain

$$u_{xx}(x,t) = \frac{1}{2}\Big(a'(t+x) + a'(t-x)\Big)$$
$$+\frac{1}{2}\Big(a(t+x)G(x,-x) - a(t-x)G(x,x)\Big)$$
$$+\frac{1}{2}\Big(r(t+x)\frac{d}{dx}G(x,-x) + r(t-x)\frac{d}{dx}G(x,x)\Big)$$

$$+\frac{1}{2}\Big(r(t+x)G_x(x,-x)+r(t-x)G_x(x,x)\Big)$$
$$+\frac{1}{2}\int_{-x}^{x} r(t-\tau)G_{xx}(x,\tau)\,d\tau.$$

Together with (2.6.35), (2.6.39) and (2.6.34) this yields

$$u_{xx}(x,t)-q(x)u(x,t)-u_{tt}(x,t)=\frac{1}{2}\int_{-x}^{x} r(t-\tau)g(x,\tau)\,d\tau, \tag{2.6.40}$$
$$-\infty<t<\infty,\ x\geq 0,$$

where

$$g(x,t)=G_{xx}(x,t)-G_{tt}(x,t)-q(x)G(x,t).$$

Let us show that

$$u(x,t)=0, \quad x>|t|. \tag{2.6.41}$$

Indeed, it follows from (2.6.36) and (2.6.21) that $u_t(x,t)=0$ for $x>|t|$, and consequently $u(x,t)\equiv C_0(x)$ for $x>|t|$. Taking $t=0$ in (2.6.35) we infer like above

$$C_0(x)=\frac{1}{2}\left(r(x)+r(-x)\right)+\frac{1}{2}\int_{-x}^{x} r(-\tau)G(x,\tau)\,d\tau=0,$$

i.e. (2.6.41) holds.

It follows from (2.6.40) and (2.6.41) that

$$\frac{1}{2}\int_{-x}^{x} r(t-\tau)g(x,\tau)\,d\tau=0, \quad x>|t|. \tag{2.6.42}$$

Differentiating (2.6.42) with respect to t and taking (2.6.20) into account we deduce

$$\frac{1}{2}\left(r(0+)g(x,t)-r(0-)g(x,t)\right)+\frac{1}{2}\int_{-x}^{x} a(t-\tau)g(x,\tau)\,d\tau=0,$$

or

$$g(x,t)+\int_{0}^{x} F(t,\tau)g(x,\tau)\,d\tau=0.$$

According to Theorem 2.6.2 this homogeneous equation has only the trivial solution $g(x,t)=0$, i.e.

$$G_{tt}=G_{xx}-q(x)G, \quad 0<|t|<x. \tag{2.6.43}$$

Furthermore, it follows from (2.6.38) for $t=0$ and (2.6.41) that

$$0=\frac{1}{2}\left(a'(x)+a'(-x)\right)+G_t(x,0)+\frac{1}{2}\int_{-x}^{x} a'(-\tau)G(x,\tau)\,d\tau.$$

Since $a'(x)=-a'(-x)$, $G(x,t)=G(x,-t)$, we infer

$$\frac{\partial G(x,t)}{\partial t}\Big|_{t=0}=0. \tag{2.6.44}$$

According to (2.6.34) the function $G(x,t)$ satisfies also (2.6.18).

It follows from (2.6.40) and (2.6.43) that

$$u_{tt}(x,t) = u_{xx}(x,t) - q(x)u(x,t), \quad -\infty < t < \infty, x \geq 0.$$

Moreover, (2.6.35) and (2.6.37) imply (with h=G(0,0))

$$u_{|x=0} = r(t), \quad u_{x|x=0} = hr(t).$$

Let us show that

$$u(x,x) = 1, \quad x \geq 0. \tag{2.6.45}$$

Since the function $G(x,t)$ satisfies (2.6.43), (2.6.44) and (2.6.18), we get according to (2.6.17),

$$\tilde{u}(x,t) = \frac{1}{2}\left(\tilde{r}(t+x) + \tilde{r}(t-x)\right) + \frac{1}{2}\int_{-x}^{x} \tilde{r}(t-\tau)G(x,\tau)\,d\tau. \tag{2.6.46}$$

Comparing (2.6.35) with (2.6.46) we get

$$\hat{u}(x,t) = \frac{1}{2}\left(\hat{r}(t+x) + \hat{r}(t-x)\right) + \frac{1}{2}\int_{-x}^{x} \hat{r}(t-\tau)G(x,\tau)\,d\tau,$$

where $\hat{u} = u - \tilde{u}$, $\hat{r} = r - \tilde{r}$. Since the function $\hat{r}(t)$ is continuous for $-\infty < t < \infty$, it follows that the function $\hat{u}(x,t)$ is also continuous for $-\infty < t < \infty$, $x > 0$. On the other hand, according to (2.6.41), $\hat{u}(x,t) = 0$ for $x > |t|$, and consequently $\hat{u}(x,x) = 0$. By (2.6.2), $\tilde{u}(x,x) = 1$, and we arrive at (2.6.45).

Thus, the function $u(x,t)$ is a solution of the boundary value problem (2.6.1)-(2.6.2). By virtue of Theorem 2.6.1 we obtain $u(x,t) = \tilde{u}(x,t)$, and consequently $r(t) = \tilde{r}(t)$. Theorem 2.6.4 is proved. □

2.7. Inverse Spectral Problems

1. Inverse problems on a finite interval. Uniqueness theorems

Let us consider the boundary value problem $L = L(q(x),h,H)$:

$$\ell y := -y'' + q(x)y = \lambda y, \quad 0 < x < \pi, \tag{2.7.1}$$

$$U(y) := y'(0) - hy(0) = 0, \quad V(y) := y'(\pi) + Hy(\pi) = 0. \tag{2.7.2}$$

Here λ is the spectral parameter; $q(x), h$ and H are real; $q(x) \in L_2(0,\pi)$. We shall subsequently refer to q as the potential. The operator ℓ is called *the Sturm-Liouville operator.* In Section 2.2 we established properties of the eigenvalues and the eigenfunctions of L. In this section we study inverse problems of spectral analysis for the Sturm-Liouville operators. Inverse spectral problems of this type consist in recovering the potential and the coefficients of the boundary conditions from the given spectral characteristics. Such problems often appear in mathematics, mechanics, physics, geophysics and other branches of natural sciences. Inverse problems also play an important role in solving nonlinear evolution equations of mathematical physics (see Section 2.9). There are close connections of inverse spectral problems and inverse problems for equations of mathematical physics; this is the reason why we study this topic below in further details. We will use the notations and facts from Section 2.2.

In this subsection we give various formulations of the inverse problems and prove the corresponding uniqueness theorems. We present several methods for proving these theorems. These methods have a wide area for applications and allow one to study various classes of inverse spectral problems.

The first result in the inverse problem theory is due to Ambarzumian [23]. Consider the boundary value problem $L(q(x),0,0)$, i.e.

$$-y''+q(x)y=\lambda y, \quad y'(0)=y'(\pi)=0. \tag{2.7.3}$$

Clearly, if $q(x)=0$ a.e. on $(0,\pi)$, then the eigenvalues of (2.7.3) have the form $\lambda_n = n^2,\ n\geq 0$. Ambarzumian proved the inverse assertion:

Theorem 2.7.1. *If the eigenvalues of* (2.7.3) *are* $\lambda_n=n^2,\ n\geq 0$, *then* $q(x)=0$ *a.e. on* $(0,\pi)$.

Proof. It follows from (2.2.49) that $\omega=0$, i.e. $\int_0^\pi q(x)\,dx=0$. Let $y_0(x)$ be an eigenfunction for the first eigenvalue $\lambda_0=0$. Then

$$y_0''(x)-q(x)y_0(x)=0, \quad y_0'(0)=y_0'(\pi)=0.$$

According to Sturm's oscillation theorem, the function $y_0(x)$ has no zeros in the interval $x\in[0,\pi]$. Taking into account the relation

$$\frac{y_0''(x)}{y_0(x)}=\Big(\frac{y_0'(x)}{y_0(x)}\Big)^2+\Big(\frac{y_0'(x)}{y_0(x)}\Big)',$$

we get

$$0=\int_0^\pi q(x)\,dx=\int_0^\pi \frac{y_0''(x)}{y_0(x)}\,dx=\int_0^\pi \Big(\frac{y_0'(x)}{y_0(x)}\Big)^2\,dx.$$

Thus, $y_0'(x)\equiv 0$, i.e. $y_0(x)\equiv const$, $q(x)=0$ a.e. on $(0,\pi)$. □

Remark 2.7.1. Actually we have proved a more general assertion than Theorem 2.7.1, namely:

If $\lambda_0=\frac{1}{\pi}\int_0^\pi q(x)\,dx$, then $q(x)=\lambda_0$ a.e. on $(0,\pi)$.

Ambarzumian‘s result is an exception from the rule. In general, the specification of the spectrum does not uniquely determine the operator. Below we provide uniqueness theorems where we point out spectral characteristics which uniquely determine the operator.

Consider the following inverse problem for $L=L(q(x),h,H)$:

Inverse Problem 2.7.1. Given the spectral data $\{\lambda_n,\alpha_n\}_{n\geq 0}$, construct the potential $q(x)$ and the coefficients h and H of the boundary conditions.

Let us prove the uniqueness theorem for the solution of Inverse Problem 2.7.1. For this purpose we agree that together with L we consider here and in the sequel a boundary value problem $\tilde{L} = L(\tilde{q}(x), \tilde{h}, \tilde{H})$ of the same form but with different coefficients. If a certain symbol γ denotes an object related to L, then the corresponding symbol $\tilde{\gamma}$ with tilde denotes the analogous object related to $\tilde{L}$, and $\hat{\gamma} := \gamma - \tilde{\gamma}$.

Theorem 2.7.2. *If $\lambda_n = \tilde{\lambda}_n$, $\alpha_n = \tilde{\alpha}_n$, $n \geq 0$, then $L = \tilde{L}$, i.e. $q(x) = \tilde{q}(x)$ a.e. on $(0,\pi)$, $h = \tilde{h}$ and $H = \tilde{H}$. Thus, the specification of the spectral data $\{\lambda_n, \alpha_n\}_{n\geq 0}$ uniquely determines the potential and the coefficients of the boundary conditions.*

We give here two proofs of Theorem 2.7.2. The first proof is due to Marchenko [15] and uses the transformation operator and Parseval‘s equality (2.2.60). The second proof is due to Levinson [24] and uses the contour integral method.

Marchenko‘s proof. Let $\varphi(x,\lambda)$ be the solution of (2.7.1) under the initial conditions $\varphi(0,\lambda) = 1$, $\varphi'(0,\lambda) = h$. In Section 2.6 we obtained the following representation for the solution $\varphi(x,\lambda)$:

$$\varphi(x,\lambda) = \cos\rho x + \int_0^x G(x,t)\cos\rho t\,dt, \tag{2.7.4}$$

where $\lambda = \rho^2$, $G(x,t)$ is a real continuous function, and (2.6.18) holds. Similarly,

$$\tilde{\varphi}(x,\lambda) = \cos\rho x + \int_0^x \tilde{G}(x,t)\cos\rho t\,dt.$$

In other words,

$$\varphi(x,\lambda) = (E+G)\cos\rho x, \quad \tilde{\varphi}(x,\lambda) = (E+\tilde{G})\cos\rho x,$$

where

$$(E+G)f(x) = f(x) + \int_0^x G(x,t)f(t)\,dt,$$

$$(E+\tilde{G})f(x) = f(x) + \int_0^x \tilde{G}(x,t)f(t)\,dt.$$

One can consider the relation $\tilde{\varphi}(x,\lambda) = (E+\tilde{G})\cos\rho x$ as a Volterra integral equation (see [14] for the theory of integral equations) with respect to $\cos\rho x$. Solving this equation we get

$$\cos\rho x = \tilde{\varphi}(x,\lambda) + \int_0^x \tilde{H}(x,t)\tilde{\varphi}(t,\lambda)\,dt,$$

where $\tilde{H}(x,t)$ is a continuous function which is the kernel of the inverse operator:

$$(E+\tilde{H}) = (E+\tilde{G})^{-1}, \quad \tilde{H}f(x) = \int_0^x \tilde{H}(x,t)f(t)\,dt.$$

Consequently

$$\varphi(x,\lambda) = \tilde{\varphi}(x,\lambda) + \int_0^x Q(x,t)\tilde{\varphi}(t,\lambda)\,dt, \tag{2.7.5}$$

where $Q(x,t)$ is a real continuous function.

Let $f(x) \in L_2(0,\pi)$. It follows from (2.7.5) that

$$\int_0^\pi f(x)\varphi(x,\lambda)\,dx = \int_0^\pi g(x)\tilde{\varphi}(x,\lambda)\,dx,$$

where

$$g(x) = f(x) + \int_x^\pi Q(t,x) f(t)\, dt.$$

Hence, for all $n \geq 0$,

$$a_n = \tilde{b}_n,$$

$$a_n := \int_0^\pi f(x)\varphi(x,\lambda_n)\, dx, \quad \tilde{b}_n := \int_0^\pi g(x)\tilde{\varphi}(x,\lambda_n)\, dx.$$

Using Parseval‘s equality (2.2.60), we calculate

$$\int_0^\pi |f(x)|^2\, dx = \sum_{n=0}^\infty \frac{|a_n|^2}{\alpha_n} = \sum_{n=0}^\infty \frac{|\tilde{b}_n|^2}{\alpha_n} = \sum_{n=0}^\infty \frac{|\tilde{b}_n|^2}{\tilde{\alpha}_n} = \int_0^\pi |g(x)|^2\, dx,$$

i.e.

$$\|f\|_{L_2} = \|g\|_{L_2}. \tag{2.7.6}$$

Consider the operator

$$Af(x) = f(x) + \int_x^\pi Q(t,x) f(t)\, dt.$$

Then $Af = g$. By virtue of (2.7.6), $\|Af\|_{L_2} = \|f\|_{L_2}$ for any $f(x) \in L_2(0,\pi)$. Consequently, $A^* = A^{-1}$, but this is possible only if $Q(x,t) \equiv 0$. Thus, $\varphi(x,\lambda) \equiv \tilde{\varphi}(x,\lambda)$, i.e. $q(x) = \tilde{q}(x)$ a.e. on $(0,\pi)$, $h = \tilde{h}$, $H = \tilde{H}$.

Levinson‘s proof. Let $f(x)$, $x \in [0,\pi]$ be an absolutely continuous function. Consider the function

$$Y^0(x,\lambda) = -\frac{1}{\Delta(\lambda)}\Big(\psi(x,\lambda)\int_0^x f(t)\tilde{\varphi}(t,\lambda)\, dt + \varphi(x,\lambda)\int_x^\pi f(t)\tilde{\psi}(t,\lambda)\, dt\Big)$$

and the contour integral

$$I_N^0(x) = \frac{1}{2\pi i}\int_{\Gamma_N} Y^0(x,\lambda)\, d\lambda.$$

The idea used here comes from the proof of Theorem 2.2.7 but here the function $Y^0(x,\lambda)$ is constructed from solutions of two boundary value problems.

Repeating the arguments of the proof of Theorem 2.2.7 we calculate

$$Y^0(x,\lambda) = \frac{f(x)}{\lambda} - \frac{Z^0(x,\lambda)}{\lambda},$$

where

$$Z^0(x,\lambda) = \frac{1}{\Delta(\lambda)}\Big\{f(x)[\varphi(x,\lambda)(\tilde{\psi}'(x,\lambda) - \psi'(x,\lambda))$$

$$-\psi(x,\lambda)(\tilde{\varphi}'(x,\lambda) - \varphi'(x,\lambda))] + \tilde{h}f(0)\psi(x,\lambda) + \tilde{H}f(\pi)\varphi(x,\lambda)$$

$$+\psi(x,\lambda)\int_0^x (\tilde{\varphi}'(t,\lambda)f'(t) + \tilde{q}(t)\tilde{\varphi}(t,\lambda)f(t))\, dt$$

$$+\varphi(x,\lambda)\int_x^\pi (\tilde{\psi}'(t,\lambda)f'(t) + \tilde{q}(t)\tilde{\psi}(t,\lambda)f(t))\, dt\Big\}.$$

The asymptotic properties for $\tilde{\varphi}(x,\lambda)$ and $\tilde{\psi}(x,\lambda)$ are the same as for $\varphi(x,\lambda)$ and $\psi(x,\lambda)$. Therefore, by similar arguments as in the proof of Theorem 2.2.7 one can obtain

$$I_N^0(x) = f(x) + \varepsilon_N^0(x), \quad \lim_{N\to\infty} \max_{0\le x\le\pi} |\varepsilon_N^0(x)| = 0.$$

On the other hand, we can calculate $I_N^0(x)$ with the help of the residue theorem:

$$I_N^0(x) = \sum_{n=0}^{N} \Big(-\frac{1}{\dot{\Delta}(\lambda_n)} \Big) \Big(\psi(x,\lambda_n) \int_0^x f(t)\tilde{\varphi}(t,\lambda_n)\, dt$$

$$+\varphi(x,\lambda_n) \int_x^\pi f(t)\tilde{\psi}(t,\lambda_n)\, dt \Big).$$

It follows from Lemma 2.2.1 and Theorem 2.2.6 that under the hypothesis of Theorem 2.7.2 we have $\beta_n = \tilde{\beta}_n$. Consequently,

$$I_N^0(x) = \sum_{n=0}^{N} \frac{1}{\alpha_n} \varphi(x,\lambda_n) \int_0^\pi f(t)\tilde{\varphi}(t,\lambda_n)\, dt.$$

If $N \to \infty$ we get

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{\alpha_n} \varphi(x,\lambda_n) \int_0^\pi f(t)\tilde{\varphi}(t,\lambda_n)\, dt.$$

Together with (2.2.59) this gives

$$\int_0^\pi f(t)(\varphi(t,\lambda_n) - \tilde{\varphi}(t,\lambda_n))\, dt = 0.$$

Since $f(x)$ is an arbitrary absolutely continuous function we conclude that $\varphi(x,\lambda_n) = \tilde{\varphi}(x,\lambda_n)$ for all $n \ge 0$ and $x \in [0,\pi]$. Consequently, $q(x) = \tilde{q}(x)$ a.e. on $(0,\pi)$, $h = \tilde{h}$, $H = \tilde{H}$. □

The Weyl function. Let $\Phi(x,\lambda)$ be the solution of (2.7.1) under the conditions $U(\Phi) = 1$, $V(\Phi) = 0$. We set $M(\lambda) := \Phi(0,\lambda)$. The functions $\Phi(x,\lambda)$ and $M(\lambda)$ are called the *Weyl solution* and the *Weyl function* (or Weyl-Titchmarsh function) for the boundary value problem L, respectively. The Weyl function was introduced first (for the case of the half-line) by H. Weyl. For further discussions on the Weyl function see, for example, [25]. Clearly,

$$\Phi(x,\lambda) = -\frac{\psi(x,\lambda)}{\Delta(\lambda)} = S(x,\lambda) + M(\lambda)\varphi(x,\lambda), \tag{2.7.7}$$

$$\langle \varphi(x,\lambda), \Phi(x,\lambda) \rangle \equiv 1, \tag{2.7.8}$$

where $S(x,\lambda)$ is the solution of (2.7.1) under the initial conditions $S(0,\lambda) = 0, S'(0,\lambda) = 1$. In particular, for $x = 0$ this yields

$$M(\lambda) = -\frac{\Delta^0(\lambda)}{\Delta(\lambda)}, \tag{2.7.9}$$

where $\Delta^0(\lambda) := \psi(0,\lambda)$ is the characteristic function of the boundary value problem $L^0 = L^0(q(x),H)$ for equation (2.7.1) with the boundary conditions $y(0) = V(y) = 0$. The eigenvalues $\{\lambda_n^0\}_{n\geq 0}$ of L^0 are simple and coincide with zeros of $\Delta^0(\lambda)$. Moreover, similarly to (2.2.57) one can get

$$\Delta^0(\lambda) = \prod_{n=0}^{\infty} \frac{\lambda_n^0 - \lambda}{(n+1/2)^2}.$$

Thus, the Weyl function is meromorphic with simple poles in $\{\lambda_n\}_{n\geq 0}$ and with simple zeros in $\{\lambda_n^0\}_{n\geq 0}$.

Theorem 2.7.3. *The following representation holds*

$$M(\lambda) = \sum_{n=0}^{\infty} \frac{1}{\alpha_n(\lambda - \lambda_n)}. \tag{2.7.10}$$

Proof. Since $\Delta^0(\lambda) = \psi(0,\lambda)$, it follows from (2.2.46) that $|\Delta^0(\lambda)| \leq C\exp(|\tau|\pi)$. Then, using (2.7.9) and (2.2.53), we get for sufficiently large $\rho^* > 0$,

$$|M(\lambda)| \leq \frac{C_\delta}{|\rho|}, \quad \rho \in G_\delta,\ |\rho| \geq \rho^*. \tag{2.7.11}$$

Further, using (2.7.9) and Lemma 2.2.1, we calculate

$$\operatorname*{Res}_{\lambda=\lambda_n} M(\lambda) = -\frac{\Delta^0(\lambda_n)}{\dot{\Delta}(\lambda_n)} = -\frac{\beta_n}{\dot{\Delta}(\lambda_n)} = \frac{1}{\alpha_n}. \tag{2.7.12}$$

Consider the contour integral

$$J_N(\lambda) = \frac{1}{2\pi i}\int_{\Gamma_N} \frac{M(\mu)}{\lambda - \mu}\, d\mu, \quad \lambda \in int\,\Gamma_N.$$

By virtue of (2.7.11), $\lim_{N\to\infty} J_N(\lambda) = 0$. On the other hand, the residue theorem and (2.7.12) yield

$$J_N(\lambda) = -M(\lambda) + \sum_{n=0}^{N} \frac{1}{\alpha_n(\lambda - \lambda_n)},$$

and Theorem 2.7.3 is proved. □

We consider the following inverse problem:

Inverse Problem 2.7.2. Given the Weyl function $M(\lambda)$, construct $L(q(x), h, H)$.

Let us prove the uniqueness theorem for Inverse Problem 2.7.2.

Theorem 2.7.4. *If $M(\lambda) = \tilde{M}(\lambda)$, then $L = \tilde{L}$. Thus, the specification of the Weyl function uniquely determines the operator.*

Proof. Let us define the matrix $P(x,\lambda) = [P_{jk}(x,\lambda)]_{j,k=1,2}$ by the formula

$$P(x,\lambda)\begin{bmatrix} \tilde{\varphi}(x,\lambda) & \tilde{\Phi}(x,\lambda) \\ \tilde{\varphi}'(x,\lambda) & \tilde{\Phi}'(x,\lambda) \end{bmatrix} = \begin{bmatrix} \varphi(x,\lambda) & \Phi(x,\lambda) \\ \varphi'(x,\lambda) & \Phi'(x,\lambda) \end{bmatrix}. \tag{2.7.13}$$

Using (2.7.8) and (2.7.13) we calculate

$$\left.\begin{aligned} P_{j1}(x,\lambda) &= \varphi^{(j-1)}(x,\lambda)\tilde{\Phi}'(x,\lambda) - \Phi^{(j-1)}(x,\lambda)\tilde{\varphi}'(x,\lambda), \\ P_{j2}(x,\lambda) &= \Phi^{(j-1)}(x,\lambda)\tilde{\varphi}(x,\lambda) - \varphi^{(j-1)}(x,\lambda)\tilde{\Phi}(x,\lambda), \end{aligned}\right\} \tag{2.7.14}$$

$$\left.\begin{aligned} \varphi(x,\lambda) &= P_{11}(x,\lambda)\tilde{\varphi}(x,\lambda) + P_{12}(x,\lambda)\tilde{\varphi}'(x,\lambda), \\ \Phi(x,\lambda) &= P_{11}(x,\lambda)\tilde{\Phi}(x,\lambda) + P_{12}(x,\lambda)\tilde{\Phi}'(x,\lambda). \end{aligned}\right\} \tag{2.7.15}$$

It follows from (2.7.14), (2.7.7) and (2.7.8) that

$$P_{11}(x,\lambda) = 1 + \frac{1}{\Delta(\lambda)}\Big(\psi(x,\lambda)(\tilde{\varphi}'(x,\lambda) - \varphi'(x,\lambda))$$

$$-\varphi(x,\lambda)(\tilde{\psi}'(x,\lambda) - \psi'(x,\lambda))\Big),$$

$$P_{12}(x,\lambda) = \frac{1}{\Delta(\lambda)}\Big(\varphi(x,\lambda)\tilde{\psi}(x,\lambda) - \psi(x,\lambda)\tilde{\varphi}(x,\lambda)\Big).$$

By virtue of (2.2.45), (2.2.46) and (2.2.53), this yields

$$|P_{11}(x,\lambda) - 1| \le \frac{C_\delta}{|\rho|}, \quad |P_{12}(x,\lambda)| \le \frac{C_\delta}{|\rho|}, \quad \rho \in G_\delta,\ |\rho| \ge \rho^*, \tag{2.7.16}$$

$$|P_{22}(x,\lambda) - 1| \le \frac{C_\delta}{|\rho|}, \quad |P_{21}(x,\lambda)| \le C_\delta, \quad \rho \in G_\delta,\ |\rho| \ge \rho^*. \tag{2.7.17}$$

According to (2.7.7) and (2.7.14),

$$P_{11}(x,\lambda) = \varphi(x,\lambda)\tilde{S}'(x,\lambda) - S(x,\lambda)\tilde{\varphi}'(x,\lambda)$$

$$+(\tilde{M}(\lambda) - M(\lambda)\varphi(x,\lambda)\tilde{\varphi}'(x,\lambda),$$

$$P_{12}(x,\lambda) = S(x,\lambda)\tilde{\varphi}(x,\lambda) - \varphi(x,\lambda)\tilde{S}(x,\lambda)$$

$$+(M(\lambda) - \tilde{M}(\lambda))\varphi(x,\lambda)\tilde{\varphi}(x,\lambda).$$

Thus, if $M(\lambda) \equiv \tilde{M}(\lambda)$, then for each fixed x, the functions $P_{11}(x,\lambda)$ and $P_{12}(x,\lambda)$ are entire in λ. Together with (2.7.16) this yields

$$P_{11}(x,\lambda) \equiv 1, \quad P_{12}(x,\lambda) \equiv 0.$$

Substituting into (2.7.15) we get

$$\varphi(x,\lambda) \equiv \tilde{\varphi}(x,\lambda), \quad \Phi(x,\lambda) \equiv \tilde{\Phi}(x,\lambda)$$

for all x and λ, and consequently, $L = \tilde{L}$. □

Remark 2.7.2. According to (2.7.10), the specification of the Weyl function $M(\lambda)$ is equivalent to the specification of the spectral data $\{\lambda_n, \alpha_n\}_{n\ge 0}$. On the other hand, by virtue of (2.7.9) zeros and poles of the Weyl function $M(\lambda)$ coincide with the spectra of the boundary value problems L and L^0, respectively. Consequently, the specification of the Weyl function $M(\lambda)$ is equivalent to the specification of two spectra $\{\lambda_n\}$ and $\{\lambda_n^0,\}$. Thus, the inverse problems of recovering the Sturm-Liouville equation from the spectral data and from two spectra are particular cases of Inverse Problem 2.7.2 of recovering the Sturm-Liouville equation from the given Weyl function, and we have several independent methods for proving the uniqueness theorems. The Weyl function is a very natural and convenient spectral characteristic in the inverse problem theory. Using the concept of the Weyl function and its generalizations we can formulate and study inverse problems for various classes of operators. For example, inverse problems of recovering higher-order differential operators and systems from the Weyl functions has been studied in [19] and [26]. We will also use the Weyl function below for the investigation of the Sturm-Liouville operator on the half-line.

2. Solution of the inverse problem on a finite interval

In this subsection we obtain an algorithm for the solution of Inverse Problem 2.7.1 and provide necessary and sufficient conditions for its solvability. For this purpose we use the so-called *transformation operator method* ([15], [16]). The central role in this method is played by a linear integral equation with respect to the kernel of the transformation operator (see Theorem 2.7.5). The main results of this subsection are stated in Theorem 2.7.6.

We first prove several auxiliary assertions.

Lemma 2.7.1. *In a Banach space B, consider the equations*

$$(E+A_0)y_0 = f_0,$$

$$(E+A)y = f,$$

where A and A_0 are linear bounded operators, acting from B to B, and E is the identity operator. Suppose that there exists the linear bounded operator $R_0 := (E+A_0)^{-1}$, this yields in particular that the equation $(E+A_0)y_0 = f_0$ is uniquely solvable in B. If

$$\|A-A_0\| \le (2\|R_0\|)^{-1},$$

then there exists the linear bounded operator $R := (E+A)^{-1}$ with

$$R = R_0\Big(E+\sum_{k=1}^{\infty}((A_0-A)R_0)^k\Big),$$

and

$$\|R-R_0\| \le 2\|R_0\|^2\,\|A-A_0\|.$$

Moreover, y and y_0 satisfy the estimate

$$\|y-y_0\| \le C_0(\|A-A_0\|+\|f-f_0\|),$$

where C_0 depends only on $\|R_0\|$ and $\|f_0\|$.

Proof. We have

$$E+A = (E+A_0)+(A-A_0) = \Big(E+(A-A_0)R_0\Big)(E+A_0).$$

Under the assumptions of the lemma it follows that $\|(A-A_0)R_0\| \le 1/2$, and consequently there exists the linear bounded operator

$$R := (E+A)^{-1} = R_0\Big(E+(A-A_0)R_0\Big)^{-1} = R_0\Big(E+\sum_{k=1}^{\infty}((A_0-A)R_0)^k\Big).$$

This yields in particular that $\|R\| \le 2\|R_0\|$. Using again the assumption on $\|A-A_0\|$ we infer

$$\|R-R_0\| \le \|R_0\| \frac{\|(A-A_0)R_0\|}{1-\|(A-A_0)R_0\|} \le 2\|R_0\|^2\|A-A_0\|.$$

Furthermore,

$$y-y_0 = Rf - R_0 f_0 = (R-R_0)f_0 + R(f-f_0).$$

Hence

$$\|y-y_0\| \le 2\|R_0\|^2\|f_0\|\|A-A_0\| + 2\|R_0\|\|f-f_0\|.$$

□

The following lemma is an obvious corollary of Lemma 2.7.1 (applied in a corresponding function space).

Lemma 2.7.2. *Consider the integral equation*

$$y(t,\alpha) + \int_a^b A(t,s,\alpha)y(s,\alpha)\,ds = f(t,\alpha), \quad a \le t \le b, \tag{2.7.18}$$

where $A(t,s,\alpha)$ and $f(t,\alpha)$ are continuous functions. Assume that for a fixed $\alpha=\alpha_0$ the homogeneous equation

$$z(t) + \int_a^b A_0(t,s)z(s)\,ds = 0, \quad A_0(t,s) := A(t,s,\alpha_0)$$

has only the trivial solution. Then in a neighbourhood of the point $\alpha=\alpha_0$, equation $(2.7.18)$ has a unique solution $y(t,\alpha)$, which is continuous with respect to t and α. Moreover, the function $y(t,\alpha)$ has the same smoothness as $A(t,s,\alpha)$ and $f(t,\alpha)$.

Lemma 2.7.3. *Let numbers $\{\rho_n,\ \alpha_n\}_{n\ge 0}$ of the form*

$$\rho_n = n + \frac{\omega}{\pi n} + \frac{\kappa_n}{n},\ \alpha_n = \frac{\pi}{2} + \frac{\kappa_{n1}}{n},\ \{\kappa_n\},\{\kappa_{n1}\} \in l_2,\ \alpha_n \ne 0 \tag{2.7.19}$$

be given. Denote

$$a(x) = \sum_{n=0}^{\infty}\Big(\frac{\cos\rho_n x}{\alpha_n} - \frac{\cos nx}{\alpha_n^0}\Big), \tag{2.7.20}$$

where

$$\alpha_n^0 = \begin{cases} \dfrac{\pi}{2}, & n>0,\\ \pi, & n=0.\end{cases}$$

Then the series $(2.7.20)$ converges absolutely and uniformly on $[0,2\pi]$, and $a(x) \in W_2^1(0,2\pi)$.

Proof. Denote $\delta_n = \rho_n - n$. Since

$$\frac{\cos\rho_n x}{\alpha_n} - \frac{\cos nx}{\alpha_n^0} = \frac{1}{\alpha_n^0}\Big(\cos\rho_n x - \cos nx\Big) + \Big(\frac{1}{\alpha_n} - \frac{1}{\alpha_n^0}\Big)\cos\rho_n x,$$

$$\cos\rho_n x - \cos nx = \cos(n+\delta_n)x - \cos nx$$

$$= -\sin\delta_n x \sin nx - 2\sin^2\frac{\delta_n x}{2}\cos nx$$

$$= -\delta_n x \sin nx - (\sin\delta_n x - \delta_n x)\sin nx - 2\sin^2\frac{\delta_n x}{2}\cos nx,$$

we have

$$a(x) = A_1(x) + A_2(x),$$

where

$$A_1(x) = -\frac{\omega x}{\pi}\sum_{n=1}^{\infty}\frac{\sin nx}{n} = -\frac{\omega x}{\pi}\cdot\frac{\pi - x}{2}, \quad 0 < x < 2\pi,$$

$$A_2(x) = \sum_{n=0}^{\infty}\Big(\frac{1}{\alpha_n} - \frac{1}{\alpha_n^0}\Big)\cos\rho_n x + \frac{1}{\pi}\Big(\cos\rho_0 x - 1\Big) - x\sum_{n=1}^{\infty}\kappa_n\frac{\sin nx}{n}$$

$$-\sum_{n=1}^{\infty}(\sin\delta_n x - \delta_n x)\sin nx - 2\sum_{n=1}^{\infty}\sin^2\frac{\delta_n x}{2}\cos nx. \tag{2.7.21}$$

Since

$$\delta_n = O\Big(\frac{1}{n}\Big), \quad \frac{1}{\alpha_n} - \frac{1}{\alpha_n^0} = \frac{\gamma_n}{n}, \quad \{\gamma_n\} \in l_2,$$

the series in (2.7.21) converge absolutely and uniformly on $[0, 2\pi]$, and $A_2(x) \in W_2^1(0, 2\pi)$. Consequently, $a(x) \in W_2^1(0, 2\pi)$. □

Now we go on to the solution of the inverse problem. Let us consider the boundary value problem $L = L(q(x), h, H)$. Let $\{\lambda_n, \alpha_n\}_{n\geq 0}$ be the spectral data of L, $\rho_n = \sqrt{\lambda_n}$. We shall solve the inverse problem of recovering L from the given spectral data $\{\lambda_n, \alpha_n\}_{n\geq 0}$. It was shown in Section 2.2 that the spectral data have the properties:

$$\rho_n = n + \frac{\omega}{\pi n} + \frac{\kappa_n}{n}, \quad \alpha_n = \frac{\pi}{2} + \frac{\kappa_{n1}}{n}, \quad \{\kappa_n\}, \{\kappa_{n1}\} \in \ell_2, \tag{2.7.22}$$

$$\alpha_n > 0, \quad \lambda_n \neq \lambda_m \ (n \neq m). \tag{2.7.23}$$

Consider the function

$$F(x,t) = \sum_{n=0}^{\infty}\Big(\frac{\cos\rho_n x\cos\rho_n t}{\alpha_n} - \frac{\cos nx\cos nt}{\alpha_n^0}\Big), \tag{2.7.24}$$

where

$$\alpha_n^0 = \begin{cases} \dfrac{\pi}{2}, & n > 0, \\ \pi, & n = 0. \end{cases}$$

Since $F(x,t) = (a(x+t) + a(x-t))/2$, then by virtue of Lemma 2.7.3, $F(x,t)$ is continuous, and $\dfrac{d}{dx}F(x,x) \in L_2(0,\pi)$.

Theorem 2.7.5. *For each fixed* $x \in (0,\pi]$, *the kernel* $G(x,t)$ *appearing in representation* $(2.7.4)$ *the linear integral equation*

$$G(x,t)+F(x,t)+\int_0^x G(x,s)F(s,t)\,ds=0, \quad 0<t<x. \tag{2.7.25}$$

This equation is the Gelfand-Levitan equation.

Thus, Theorem 2.7.5 allows one to reduce our inverse problem to the solution of the Gelfand-Levitan equation (2.7.25). We note that (2.7.25) is a Fredholm type integral equation in which x is a parameter.

Proof. One can consider the relation (2.7.4) as a Volterra integral equation with respect to $\cos\rho x$. Solving this equation we obtain

$$\cos\rho x=\varphi(x,\lambda)+\int_0^x H(x,t)\varphi(t,\lambda)\,dt, \tag{2.7.26}$$

where $H(x,t)$ is a continuous function. Using (2.7.4) and (2.7.26) we calculate

$$\sum_{n=0}^{N}\frac{\varphi(x,\lambda_n)\cos\rho_n t}{\alpha_n}=\sum_{n=0}^{N}\Big(\frac{\cos\rho_n x\cos\rho_n t}{\alpha_n}+\frac{\cos\rho_n t}{\alpha_n}\int_0^x G(x,s)\cos\rho_n s\,ds\Big),$$

$$\sum_{n=0}^{N}\frac{\varphi(x,\lambda_n)\cos\rho_n t}{\alpha_n}=\sum_{n=0}^{N}\Big(\frac{\varphi(x,\lambda_n)\varphi(t,\lambda_n)}{\alpha_n}+\frac{\varphi(x,\lambda_n)}{\alpha_n}\int_0^t H(t,s)\varphi(s,\lambda_n)\,ds\Big).$$

This yields

$$\Phi_N(x,t)=I_{N1}(x,t)+I_{N2}(x,t)+I_{N3}(x,t)+I_{N4}(x,t),$$

where

$$\Phi_N(x,t)=\sum_{n=0}^{N}\Big(\frac{\varphi(x,\lambda_n)\varphi(t,\lambda_n)}{\alpha_n}-\frac{\cos nx\cos nt}{\alpha_n^0}\Big),$$

$$I_{N1}(x,t)=\sum_{n=0}^{N}\Big(\frac{\cos\rho_n x\cos\rho_n t}{\alpha_n}-\frac{\cos nx\cos nt}{\alpha_n^0}\Big),$$

$$I_{N2}(x,t)=\sum_{n=0}^{N}\frac{\cos nt}{\alpha_n^0}\int_0^x G(x,s)\cos ns\,ds,$$

$$I_{N3}(x,t)=\sum_{n=0}^{N}\int_0^x G(x,s)\Big(\frac{\cos\rho_n t\cos\rho_n s}{\alpha_n}-\frac{\cos nt\cos ns}{\alpha_n^0}\Big)\,ds,$$

$$I_{N4}(x,t)=-\sum_{n=0}^{N}\frac{\varphi(x,\lambda_n)}{\alpha_n}\int_0^t H(t,s)\varphi(s,\lambda_n)\,ds.$$

Let $f(x)\in AC[0,\pi]$. According to Theorem 2.2.7,

$$\lim_{N\to\infty}\max_{0\le x\le\pi}\int_0^\pi f(t)\Phi_N(x,t)\,dt=0;$$

furthermore, uniformly with respect to $x \in [0,\pi]$,

$$\lim_{N\to\infty} \int_0^\pi f(t) I_{N1}(x,t)\,dt = \int_0^\pi f(t)F(x,t)\,dt,$$

$$\lim_{N\to\infty} \int_0^\pi f(t) I_{N2}(x,t)\,dt = \int_0^x f(t)G(x,t)\,dt,$$

$$\lim_{N\to\infty} \int_0^\pi f(t) I_{N3}(x,t)\,dt = \int_0^\pi f(t)\Big(\int_0^x G(x,s)F(s,t)\,ds\Big)\,dt,$$

$$\lim_{N\to\infty} \int_0^\pi f(t) I_{N4}(x,t)\,dt$$

$$= -\lim_{N\to\infty} \sum_{n=0}^N \frac{\varphi(x,\lambda_n)}{\alpha_n} \int_0^\pi \varphi(s,\lambda_n)\Big(\int_s^\pi H(t,s)f(t)\,dt\Big)\,ds = -\int_x^\pi f(t)H(t,x)\,dt.$$

Extend $G(x,t) = H(x,t) = 0$ for $x < t$. Then, in view of the arbitrariness of $f(x)$, we derive

$$G(x,t) + F(x,t) + \int_0^x G(x,s)F(s,t)\,ds - H(t,x) = 0.$$

For $t < x$, this yields (2.7.25). □

The next theorem, which is the main result of this subsection, gives us an algorithm for the solution of the inverse problem as well as necessary and sufficient conditions for its solvability.

Theorem 2.7.6. *For real numbers* $\{\lambda_n, \alpha_n\}_{n\ge 0}$ *to be the spectral data for a certain boundary value problem* $L(q(x),h,H)$ *with* $q(x) \in L_2(0,\pi)$, *it is necessary and sufficient that the relations* $(2.7.22)-(2.7.23)$ *hold. The boundary value problem* $L(q(x),h,H)$ *can be constructed by the following algorithm:*

Algorithm 2.7.1. (i) *From the given numbers* $\{\lambda_n, \alpha_n\}_{n\ge 0}$ *construct the function* $F(x,t)$ *by* $(2.7.24)$.

(ii) *Find the function* $G(x,t)$ *by solving equation* $(2.7.25)$.

(iii) *Calculate* $q(x)$, h *and* H *by the formulae*

$$q(x) = 2\frac{d}{dx}G(x,x),\ h = G(0,0),\ H = \omega - h - \frac{1}{2}\int_0^\pi q(t)\,dt. \tag{2.7.27}$$

The necessity part of Theorem 2.7.6 was proved above, here we prove the sufficiency. Let real numbers $\{\lambda_n, \alpha_n\}_{n\ge 0}$ of the form (2.7.22)-(2.7.23) be given. We construct the function $F(x,t)$ by (2.7.24) and consider equation (2.7.25).

Lemma 2.7.4. *For each fixed* $x \in (0,\pi]$, *equation* $(2.7.25)$ *has a unique solution* $G(x,t)$ *in* $L_2(0,x)$.

Proof. Since (2.7.25) is a Fredholm equation it is sufficient to prove that the homogeneous equation

$$g(t) + \int_0^x F(s,t)g(s)\,ds = 0 \tag{2.7.28}$$

has only the trivial solution $g(t) = 0$.

Let $g(t)$ be a solution of (2.7.28). Then

$$\int_0^x g^2(t)\,dt + \int_0^x\int_0^x F(s,t)g(s)g(t)\,dsdt = 0$$

or

$$\int_0^x g^2(t)\,dt + \sum_{n=0}^{\infty}\frac{1}{\alpha_n}\Big(\int_0^x g(t)\cos\rho_n t\,dt\Big)^2 - \sum_{n=0}^{\infty}\frac{1}{\alpha_n^0}\Big(\int_0^x g(t)\cos nt\,dt\Big)^2 = 0.$$

Using Parseval's equality

$$\int_0^x g^2(t)\,dt = \sum_{n=0}^{\infty}\frac{1}{\alpha_n^0}\Big(\int_0^x g(t)\cos nt\,dt\Big)^2,$$

for the function $g(t)$, extended by zero for $t > x$, we obtain

$$\sum_{n=0}^{\infty}\frac{1}{\alpha_n}\Big(\int_0^x g(t)\cos\rho_n t\,dt\Big)^2 = 0.$$

Since $\alpha_n > 0$, then

$$\int_0^x g(t)\cos\rho_n t\,dt = 0, \quad n \ge 0.$$

The system of functions $\{\cos\rho_n t\}_{n\ge 0}$ is complete in $L_2(0,\pi)$ (see Levinson's theorem [27, p.118]). This yields $g(t) = 0$. □

Let us return to the proof of Theorem 2.7.6. Let $G(x,t)$ be the solution of (2.7.25). The substitution $t \to tx$, $s \to sx$ in (2.7.25) yields

$$F(x,xt) + G(x,xt) + x\int_0^1 G(x,xs)F(xt,xs)\,ds = 0, \quad 0 \le t \le 1. \tag{2.7.29}$$

It follows from (2.7.25), (2.7.29) and Lemma 2.7.2 that the function $G(x,t)$ is continuous, and has the same smoothness as $F(x,t)$. In particular, $\frac{d}{dx}G(x,x) \in L_2(0,\pi)$.

We construct $\varphi(x,\lambda)$ by (2.7.4) and the boundary value problem $L(q(x),h,H)$ by (2.7.27). Then one can verify that $\varphi(x,\lambda)$ satisfies (2.7.1), and the numbers $\{\lambda_n,\alpha_n\}_{n\ge 0}$ are the spectral data for the constructed problem $L(q(x),h,H)$ (see [17, Sect.1.6]). Thus, Theorem 2.7.6 is proved. □

Example 2.7.1. Let $\lambda_n = n^2\ (n \ge 0)$, $\alpha_n = \frac{\pi}{2}\ (n \ge 1)$, and let $\alpha_0 > 0$ be an arbitrary positive number. Denote $a := \frac{1}{\alpha_0} - \frac{1}{\pi}$. Let us use Algorithm 2.7.1:

1) By (2.7.24), $F(x,t) \equiv a$.

2) Solving equation (2.7.25) we get easily

$$G(x,t) = -\frac{a}{1+ax}.$$

3) By (2.7.27),

$$q(x) = \frac{2a^2}{(1+ax)^2}, \quad h = -a, \quad H = \frac{a}{1+a\pi} = \frac{a\alpha_0}{\pi}.$$

By (2.7.4),

$$\varphi(x,\lambda) = \cos\rho x - \frac{a}{1+ax}\frac{\sin\rho x}{\rho}.$$

3. Sturm-Liouville operators on the half-line

In subsections 3-5 we present an introduction to the inverse problem theory for Sturm-Liouville operators on the half-line. First nonselfadjoint operators with integrable complex-valued potentials are considered. We introduce and study the Weyl function as the main spectral characteristic, prove an expansion theorem and solve the inverse problem of recovering the Sturm-Liouville operator from its Weyl function. For this purpose we use ideas of the contour integral method and the method of spectral mappings presented in [17], [18]. Moreover connections with the transformation operator method are established. Then locally integrable complex-valued potentials are studied. In this case the generalized Weyl function is introduced as the main spectral characteristic. We prove an expansion theorem and solve the inverse problem of recovering the Sturm-Liouville operator from its generalized Weyl function.

We consider the differential equation and the linear form $L = L(q(x),h)$:

$$\ell y := -y'' + q(x)y = \lambda y, \quad x > 0, \tag{2.7.30}$$

$$U(y) := y'(0) - hy(0), \tag{2.7.31}$$

where $q(x) \in L(0,\infty)$ is a complex-valued function, and h is a complex number. Let $\lambda = \rho^2$, $\rho = \sigma + i\tau$, and let for definiteness $\tau := Im\,\rho \geq 0$. Denote by Π the λ-plane with the cut $\lambda \geq 0$, and $\Pi_1 = \overline{\Pi}\setminus\{0\}$; notice that here Π and Π_1 must be considered as subsets of the Riemann surface of the squareroot-function. Then, under the map $\rho \to \rho^2 = \lambda$, Π_1 corresponds to the domain $\Omega = \{\rho :\ Im\,\rho \geq 0, \rho \neq 0\}$. Put $\Omega_\delta = \{\rho :\ Im\,\rho \geq 0, |\rho| \geq \delta\}$. Denote by W_N the set of functions $f(x), x \geq 0$ such that the functions $f^{(j)}(x)$, $j = \overline{0,N-1}$ are absolutely continuous on $[0,T]$ for each fixed $T > 0$, and $f^{(j)}(x) \in L(0,\infty)$, $j = \overline{0,N}$.

Jost and Birkhoff solutions. Let us construct a special fundamental system of solutions for equation (2.7.30) in Ω having asymptotic behavior at infinity like $\exp(\pm i\rho x)$.

Theorem 2.7.7. *Equation* (2.7.30) *has a unique solution* $y = e(x,\rho), \rho \in \Omega, x \geq 0,$ *satisfying the integral equation*

$$e(x,\rho) = \exp(i\rho x) - \frac{1}{2i\rho}\int_x^\infty (\exp(i\rho(x-t)) - \exp(i\rho(t-x)))q(t)e(t,\rho)\,dt. \tag{2.7.32}$$

The function $e(x,\rho)$ *has the following properties:*
(i_1) *For* $x \to \infty$, $\nu = 0,1,$ *and each fixed* $\delta > 0$,

$$e^{(\nu)}(x,\rho) = (i\rho)^\nu \exp(i\rho x)(1 + o(1)), \tag{2.7.33}$$

uniformly in Ω_δ*. For* $Im\,\rho > 0$, $e(x,\rho) \in L_2(0,\infty)$*. Moreover,* $e(x,\rho)$ *is the unique solution of* (2.7.30)(*up to a multiplicative constant*) *having this property.*
(i_2) *For* $|\rho| \to \infty$, $\rho \in \Omega$, $\nu = 0,1$,

$$e^{(\nu)}(x,\rho) = (i\rho)^\nu \exp(i\rho x)\Big(1 + \frac{\omega(x)}{i\rho} + o\Big(\frac{1}{\rho}\Big)\Big),\ \omega(x) := -\frac{1}{2}\int_x^\infty q(t)\,dt, \qquad (2.7.34)$$

uniformly for $x \ge 0$.
(i_3) *For each fixed* $x \ge 0$, *and* $\nu = 0,1$, *the functions* $e^{(\nu)}(x,\rho)$ *are analytic for* $Im\,\rho > 0$, *and are continuous for* $\rho \in \Omega$.
(i_4) *For real* $\rho \ne 0$, *the functions* $e(x,\rho)$ *and* $e(x,-\rho)$ *form a fundamental system of solutions for* (2.7.30)*, and*

$$\langle e(x,\rho), e(x,-\rho)\rangle = -2i\rho, \qquad (2.7.35)$$

where $\langle y,z\rangle := yz' - y'z$ *is the Wronskian.*
The function $e(x,\rho)$ *is called the* Jost solution *for* (2.7.30)*.*

Proof. We transform (2.7.32) by means of the replacement

$$e(x,\rho) = \exp(i\rho x) z(x,\rho) \qquad (2.7.36)$$

to the equation

$$z(x,\rho) = 1 - \frac{1}{2i\rho}\int_x^\infty (1 - \exp(2i\rho(t-x)))\, q(t) z(t,\rho)\,dt,\ \ge 0,\ \rho \in \Omega. \qquad (2.7.37)$$

The method of successive approximations gives

$$z_0(x,\rho) = 1,\ z_{k+1}(x,\rho) = -\frac{1}{2i\rho}\int_x^\infty (1 - \exp(2i\rho(t-x)))\, q(t) z_k(t,\rho)\,dt, \qquad (2.7.38)$$

$$z(x,\rho) = \sum_{k=0}^\infty z_k(x,\rho). \qquad (2.7.39)$$

Let us show by induction that

$$|z_k(x,\rho)| \le \frac{(Q_0(x))^k}{|\rho|^k k!}, \quad \rho \in \Omega,\ x \ge 0, \qquad (2.7.40)$$

where

$$Q_0(x) := \int_x^\infty |q(t)|\,dt.$$

Indeed, for $k = 0$, (2.7.40) is obvious. Suppose that (2.7.40) is valid for a certain fixed $k \ge 0$. Since $|1 - \exp(2i\rho(t-x))| \le 2$, (2.7.38) implies

$$|z_{k+1}(x,\rho)| \le \frac{1}{|\rho|}\int_x^\infty |q(t) z_k(t,\rho)|\,dt. \qquad (2.7.41)$$

Substituting (2.7.40) into the right-hand side of (2.7.41) we calculate

$$|z_{k+1}(x,\rho)| \le \frac{1}{|\rho|^{k+1} k!}\int_x^\infty |q(t)| (Q_0(t))^k\,dt = \frac{(Q_0(t))^{k+1}}{|\rho|^{k+1}(k+1)!}.$$

It follows from (2.7.40) that the series (2.7.39) converges absolutely for $x \geq 0,\ \rho \in \Omega$, and the function $z(x,\rho)$ is the unique solution of the integral equation (2.7.37). Moreover, by virtue of (2.7.39) and (2.7.40),

$$|z(x,\rho)| \leq \exp(Q_0(x)/|\rho|),\ |z(x,\rho)-1| \leq (Q_0(x)/|\rho|)\exp(Q_0(x)/|\rho|). \qquad (2.7.42)$$

In particular, (2.7.42) yields for each fixed $\delta > 0$,

$$z(x,\rho) = 1 + o(1), \quad x \to \infty, \qquad (2.7.43)$$

uniformly in Ω_δ, and

$$z(x,\rho) = 1 + O\Big(\frac{1}{\rho}\Big), \quad |\rho| \to \infty,\ \rho \in \Omega, \qquad (2.7.44)$$

uniformly for $x \geq 0$. Substituting (2.7.44) into the right-hand side of (2.7.37) we obtain

$$z(x,\rho) = 1 - \frac{1}{2i\rho}\int_x^\infty (1-\exp(2i\rho(t-x)))\,q(t)\,dt + O\Big(\frac{1}{\rho^2}\Big),\ |\rho| \to \infty, \qquad (2.7.45)$$

uniformly for $x \geq 0$.

Lemma 2.7.5. *Let* $q(x) \in L(0,\infty)$, *and denote*

$$J_q(x,\rho) := \int_x^\infty q(t)\exp(2i\rho(t-x))\,dt, \quad \rho \in \Omega. \qquad (2.7.46)$$

Then

$$\lim_{|\rho|\to\infty} \sup_{x\geq 0} |J_q(x,\rho)| = 0. \qquad (2.7.47)$$

Proof. 1) First we assume that $q(x) \in W_1$. Then integration by parts in (2.7.46) yields

$$J_q(x,\rho) = -\frac{q(x)}{2i\rho} - \frac{1}{2i\rho}\int_x^\infty q'(t)\exp(2i\rho(t-x))\,dt,$$

and consequently

$$\sup_{x\geq 0}|J_q(x,\rho)| \leq \frac{C_q}{|\rho|}.$$

2) Let now $q(x) \in L(0,\infty)$. Fix $\varepsilon > 0$ and choose $q_\varepsilon(x) \in W_1$ such that

$$\int_0^\infty |q(t)-q_\varepsilon(t)|\,dt < \frac{\varepsilon}{2}.$$

Then

$$|J_q(x,\rho)| \leq |J_{q_\varepsilon}(x,\rho)| + |J_{q-q_\varepsilon}(x,\rho)| \leq \frac{C_{q_\varepsilon}}{|\rho|} + \frac{\varepsilon}{2}.$$

Hence, there exists $\rho^0 > 0$ such that $\sup_{x\geq 0}|J_q(x,\rho)| \leq \varepsilon$ for $|\rho| \geq \rho^0,\ \rho \in \Omega$. By virtue of arbitrariness of $\varepsilon > 0$ we arrive at (2.7.47). $\square$

Let us return to the proof of Theorem 2.7.7. It follows from (2.7.45) and Lemma 2.7.5 that

$$z(x,\rho)=1+\frac{\omega(x)}{i\rho}+o\Big(\frac{1}{\rho}\Big),\quad |\rho|\to\infty,\ \rho\in\Omega, \tag{2.7.48}$$

uniformly for $x\ge 0.$ From (2.7.36), (2.7.38)-(2.7.40), (2.7.43) and (2.7.48) we derive $(i_1)-(i_3)$ for $\nu=0.$ Furthermore, (2.7.32) and (2.7.36) imply

$$e'(x,\rho)=(i\rho)\exp(i\rho x)\left(1-\frac{1}{2i\rho}\int_x^\infty(1+\exp(2i\rho(t-x)))\,q(t)z(t,\rho)\,dt\right). \tag{2.7.49}$$

Using (2.7.49) we get $(i_1)-(i_3)$ for $\nu=1.$ It is easy to verify by differentiation that the function $e(x,\rho)$ is a solution of (2.7.30). For real $\rho\neq 0,$ the functions $e(x,\rho)$ and $e(x,-\rho)$ satisfy (2.7.30), and by virtue of (2.7.33), $\lim\limits_{x\to\infty}\langle e(x,\rho),e(x,-\rho)\rangle=-2i\rho.$ Since the Wronskian $\langle e(x,\rho),e(x,-\rho)\rangle$ does not depend on $x,$ we arrive at (2.7.35). Lemma 2.7.5 is proved. □

Remark 2.7.3. If $q(x)\in W_N,$ then there exist functions $\omega_{s\nu}(x)$ such that for $\rho\to\infty,\ \rho\in\Omega,\ \nu=0,1,2,$

$$e^{(\nu)}(x,\rho)=(i\rho)^\nu\exp(i\rho x)\Big(1+\sum_{s=1}^{N+1}\frac{\omega_{s\nu}(x)}{(i\rho)^s}+o\Big(\frac{1}{\rho^{N+1}}\Big)\Big),\ \ \omega_{1\nu}(x)=\omega(x). \tag{2.7.50}$$

Indeed, let $q(x)\in W_1.$ Substituting (2.7.48) into the right-hand side of (2.7.37) we get

$$z(x,\rho)=1-\frac{1}{2i\rho}\int_x^\infty(1-\exp(2i\rho(t-x)))\,q(t)\,dt$$

$$-\frac{1}{2(i\rho)^2}\int_x^\infty(1-\exp(2i\rho(t-x)))\,q(t)\omega(t)\,dt+o\Big(\frac{1}{\rho^2}\Big),\quad |\rho|\to\infty.$$

Integrating by parts and using Lemma 2.7.5, we obtain

$$z(x,\rho)=1+\frac{\omega(x)}{i\rho}+\frac{\omega_{20}(x)}{(i\rho)^2}+o\Big(\frac{1}{\rho^2}\Big),\quad |\rho|\to\infty,\ \rho\in\Omega, \tag{2.7.51}$$

where

$$\omega_{20}(x)=-\frac14 q(x)+\frac14\int_x^\infty q(t)\Big(\int_t^\infty q(s)\,ds\Big)\,dt$$
$$=-\frac14 q(x)+\frac18\Big(\int_x^\infty q(t)\,dt\Big)^2.$$

By virtue of (2.7.36) and (2.7.51), we arrive at (2.7.50) for $N=1,\ \nu=0.$ Using induction one can prove (2.7.50) for all $N.$

If we additionally assume that $xq(x)\in L(0,\infty),$ then the Jost solution $e(x,\rho)$ exists also for $\rho=0.$ More precisely, the following theorem is valid.

Theorem 2.7.8. *Let $(1+x)q(x)\in L(0,\infty).$ Then the functions $e^{(\nu)}(x,\rho),\ \nu=0,1$ are continuous for $Im\,\rho\ge 0,\ x\ge 0,$ and*

$$|e(x,\rho)\exp(-i\rho x)|\le\exp(Q_1(x)), \tag{2.7.52}$$

$$|e(x,\rho)\exp(-i\rho x)-1| \le \Big(Q_1(x)-Q_1\Big(x+\frac{1}{|\rho|}\Big)\Big)\exp(Q_1(x)), \tag{2.7.53}$$

$$|e'(x,\rho)\exp(-i\rho x)-i\rho| \le Q_0(x)\exp(Q_1(x)), \tag{2.7.54}$$

where

$$Q_1(x) := \int_x^\infty Q_0(t)\,dt = \int_x^\infty (t-x)|q(t)|\,dt.$$

First we prove an auxiliary assertion.

Lemma 2.7.6. *Assume that* $c_1 \ge 0,\ u(x) \ge 0,\ v(x) \ge 0,\ (a \le x \le T \le \infty),\ u(x)$ *is bounded, and* $(x-a)v(x) \in L(a,T)$. *If*

$$u(x) \le c_1 + \int_x^T (t-x)v(t)u(t)\,dt, \tag{2.7.55}$$

then

$$u(x) \le c_1 \exp\Big(\int_x^T (t-x)v(t)\,dt\Big). \tag{2.7.56}$$

Proof. Denote

$$\xi(x) = c_1 + \int_x^T (t-x)v(t)u(t)\,dt.$$

Then

$$\xi(T) = c_1, \quad \xi'(x) = -\int_x^T v(t)u(t)\,dt, \quad \xi'' = v(x)u(x),$$

and (2.7.55) yields

$$0 \le \xi''(x) \le \xi(x)v(x).$$

Let $c_1 > 0$. Then $\xi(x) > 0$, and

$$\frac{\xi''(x)}{\xi(x)} \le v(x).$$

Hence

$$\Big(\frac{\xi'(x)}{\xi(x)}\Big)' \le v(x) - \Big(\frac{\xi'(x)}{\xi(x)}\Big)^2 \le v(x).$$

Integrating this inequality twice we get

$$-\frac{\xi'(x)}{\xi(x)} \le \int_x^T v(t)\,dt, \quad \ln\frac{\xi(x)}{\xi(T)} \le \int_x^T (t-x)v(t)\,dt,$$

and consequently,

$$\xi(x) \le c_1 \exp\Big(\int_x^T (t-x)v(t)\,dt\Big).$$

According to (2.7.55), $u(x) \le \xi(x)$, and we arrive at (2.7.56).

If $c_1 = 0$, then $\xi(x) = 0$. Indeed, suppose on the contrary that $\xi(x) \ne 0$. Since $\xi(x) \ge 0$, $\xi'(x) \le 0$, there exists $T_0 \le T$ such that $\xi(x) > 0$ for $x < T_0$, and $\xi(x) \equiv 0$ for $x \in [T_0, T]$. Repeating these arguments we get for $x < T_0$ and sufficiently small $\varepsilon > 0$,

$$\ln\frac{\xi(x)}{\xi(T_0-\varepsilon)} \le \int_x^{T_0-\varepsilon} (t-x)v(t)\,dt \le \int_x^{T_0} (t-x)v(t)\,dt,$$

which is impossible. Thus, $\xi(x)\equiv 0,$ and (2.7.56) becomes obvious. □

Proof of Theorem 2.7.8. For $Im\rho\ge 0$ we have

$$\Big|\frac{\sin\rho}{\rho}\exp(i\rho)\Big|\le 1. \tag{2.7.57}$$

Indeed, (2.7.57) is obvious for real ρ and for $|\rho|\ge 1,\ Im\rho\ge 0.$ Then, by the maximum principle [14, p.128], (2.7.57) is also valid for $|\rho|\le 1,\ Im\rho\ge 0.$

It follows from (2.7.57) that

$$\Big|\frac{1-\exp(2i\rho x)}{2i\rho}\Big|\le x \ \text{ for } Im\rho\ge 0,\ x\ge 0. \tag{2.7.58}$$

Using (2.7.38) and (2.7.58) we infer

$$|z_{k+1}(x,\rho)|\le\int_x^\infty t|q(t)z_k(t,\rho)|\,dt,\quad k\ge 0,\ Im\rho\ge 0,\ x\ge 0,$$

and consequently by induction

$$|z_k(x,\rho)|\le\frac{1}{k!}\Big(\int_x^\infty t|q(t)|\,dt\Big)^k,\quad k\ge 0,\ Im\rho\ge 0,\ x\ge 0.$$

Then, the series (2.7.39) converges absolutely and uniformly for $Im\rho\ge 0,\ x\ge 0,$ and the function $z(x,\rho)$ is continuous for $Im\rho\ge 0,\ x\ge 0.$ Moreover,

$$|z(x,\rho)|\le\exp\Big(\int_x^\infty t|q(t)|\,dt\Big),\quad Im\rho\ge 0,\ x\ge 0. \tag{2.7.59}$$

Using (2.7.36) and (2.7.49) we conclude that the functions $e^{(\nu)}(x,\rho),\ \nu=0,1$ are continuous for $Im\rho\ge 0,\ x\ge 0.$

Furthermore, it follows from (2.7.37) and (2.7.58) that

$$|z(x,\rho)|\le 1+\int_x^\infty (t-x)|q(t)z(t,\rho)|\,dt,\quad Im\rho\ge 0,\ x\ge 0.$$

By virtue of Lemma 2.7.6, this implies

$$|z(x,\rho)|\le\exp(Q_1(x)),\quad Im\rho\ge 0,\ x\ge 0, \tag{2.7.60}$$

i.e. (2.7.52) is valid. We note that (2.7.60) is more precise than (2.7.59).

Using (2.7.37), (2.7.58) and (2.7.60) we calculate

$$|z(x,\rho)-1|\le\int_x^\infty (t-x)|q(t)|\exp(Q_1(t))\,dt$$

$$\le\exp(Q_1(x))\int_x^\infty (t-x)|q(t)|\,dt,$$

and consequently,

$$|z(x,\rho)-1|\le Q_1(x)\exp(Q_1(x)),\quad Im\rho\ge 0,\ x\ge 0. \tag{2.7.61}$$

More precisely,

$$|z(x,\rho)-1| \le \int_x^{x+\frac{1}{|\rho|}} (t-x)|q(t)|\exp(Q_1(t))\,dt + \frac{1}{|\rho|}\int_{x+\frac{1}{|\rho|}}^{\infty} |q(t)|\exp(Q_1(t))\,dt$$

$$\le \exp(Q_1(x))\Big(\int_x^{\infty}(t-x)|q(t)|\,dt - \int_{x+\frac{1}{|\rho|}}^{\infty}\Big(t-x-\frac{1}{|\rho|}\Big)|q(t)|\,dt\Big)$$

$$= \Big(Q_1(x)-Q_1\Big(x+\frac{1}{|\rho|}\Big)\Big)\exp(Q_1(x)),$$

i.e. (2.7.53) is valid. At last, from (2.7.49) and (2.7.60) we obtain

$$|e'(x,\rho)\exp(-i\rho x)-i\rho| \le \int_x^{\infty}|q(t)|\exp(Q_1(t))\,dt \le \exp(Q_1(x))\int_x^{\infty}|q(t)|\,dt,$$

and we arrive at (2.7.54). Theorem 2.7.8 is proved. □

Remark 2.7.4. Consider the function

$$q(x) = \frac{2a^2}{(1+ax)^2},$$

where a is a complex number such that $a \notin (-\infty,0]$. Then $q(x)\in L(0,\infty)$, but $xq(x)\notin L(0,\infty)$. The Jost solution has in this case the form (see Example 2.7.3)

$$e(x,\rho) = \exp(i\rho x)\Big(1-\frac{a}{i\rho(1+ax)}\Big),$$

i.e. $e(x,\rho)$ has a singularity at $\rho=0$, hence we cannot omit the integrability condition in Theorem 2.7.8.

Theorem 2.7.9. *Let $(1+x)q(x)\in L(0,\infty)$. Then the Jost solution $e(x,\rho)$ can be represented in the form*

$$e(x,\rho) = \exp(i\rho x) + \int_x^{\infty} A(x,t)\exp(i\rho t)\,dt, \quad Im\,\rho \ge 0,\ x\ge 0, \tag{2.7.62}$$

where $A(x,t)$ is a continuous function for $0\le x\le t<\infty$, and

$$A(x,x) = \frac{1}{2}\int_x^{\infty} q(t)\,dt. \tag{2.7.63}$$

$$|A(x,t)| \le \frac{1}{2}Q_0\Big(\frac{x+t}{2}\Big)\exp\Big(Q_1(x)-Q_1\Big(\frac{x+t}{2}\Big)\Big), \tag{2.7.64}$$

$$1+\int_x^{\infty}|A(x,t)|\,dt \le \exp(Q_1(x)),\ \int_x^{\infty}|A(x,t)|\,dt \le Q_1(x)\exp(Q_1(x)). \tag{2.7.65}$$

Moreover, the function $A(x_1,x_2)$ has first derivatives $\dfrac{\partial A}{\partial x_i}$, $i=1,2;$ the functions

$$\frac{\partial A(x_1,x_2)}{\partial x_i} + \frac{1}{4}q\Big(\frac{x_1+x_2}{2}\Big)$$

are absolutely continuous with respect to x_1 *and* x_2, *and satisfy the estimates*

$$\Big|\frac{\partial A(x_1,x_2)}{\partial x_i}+\frac{1}{4}q\Big(\frac{x_1+x_2}{2}\Big)\Big|$$

$$\le\frac{1}{2}Q_0(x_1)Q_0\Big(\frac{x_1+x_2}{2}\Big)\exp\Big(Q_1(x_1)-Q_1\Big(\frac{x_1+x_2}{2}\Big)\Big),\quad i=1,2. \tag{2.7.66}$$

Proof. According to (2.7.36) and (2.7.39) we have

$$e(x,\rho)=\sum_{k=0}^{\infty}\varepsilon_k(x,\rho),\quad \varepsilon_k(x,\rho)=z_k(x,\rho)\exp(i\rho x). \tag{2.7.67}$$

Let us show by induction that the following representation is valid

$$\varepsilon_k(x,\rho)=\int_x^\infty a_k(x,t)\exp(i\rho t)\,dt,\quad k\ge 1, \tag{2.7.68}$$

where the functions $a_k(x,t)$ do not depend on ρ.

First we calculate $\varepsilon_1(x,\rho)$. By virtue of (2.7.38) and (2.7.67),

$$\varepsilon_1(x,\rho)=\int_x^\infty\frac{\sin\rho(s-x)}{\rho}\exp(i\rho s)q(s)\,ds$$

$$=\frac{1}{2}\int_x^\infty q(s)\Big(\int_x^{2s-x}\exp(i\rho t)\,dt\Big)\,ds.$$

Interchanging the order of integration we obtain that (2.7.68) holds for $k=1$, where

$$a_1(x,t)=\frac{1}{2}\int_{(t+x)/2}^\infty q(s)\,ds.$$

Suppose now that (2.7.68) is valid for a certain $k\ge 1$. Then

$$\varepsilon_{k+1}(x,\rho)=\int_x^\infty\frac{\sin\rho(s-x)}{\rho}q(s)\varepsilon_k(s,\rho)\,ds$$

$$=\int_x^\infty\frac{\sin\rho(s-x)}{\rho}q(s)\Big(\int_s^\infty a_k(s,u)\exp(i\rho u)\,du\Big)\,ds$$

$$=\frac{1}{2}\int_x^\infty q(s)\Big(\int_s^\infty a_k(s,u)\Big(\int_{-s+u+x}^{s+u-x}\exp(i\rho t)\,dt\Big)\,du\Big)\,ds.$$

We extend $a_k(s,u)$ by zero for $u<s$. For $s\ge x$ this yields

$$\int_s^\infty a_k(s,u)\Big(\int_{-s+u+x}^{s+u-x}\exp(i\rho t)\,dt\Big)\,du=\int_x^\infty\exp(i\rho t)\Big(\int_{t-s+x}^{t+s-x}a_k(s,u)\,du\Big)\,dt.$$

Therefore

$$\varepsilon_{k+1}(x,\rho)=\frac{1}{2}\int_x^\infty\exp(i\rho t)\Big(\int_x^\infty q(s)\Big(\int_{t-s+x}^{t+s-x}a_k(s,u)\,du\Big)\,ds\Big)\,dt$$

$$= \int_x^\infty a_{k+1}(x,t)\exp(i\rho t)\,dt,$$

where

$$a_{k+1}(x,t) = \frac{1}{2}\int_x^\infty q(s)\Big(\int_{t-s+x}^{t+s-x} a_k(s,u)\,du\Big)\,ds, \quad t \ge x.$$

Changing the variables according to $u+s=2\alpha,\ u-s=2\beta,$ we obtain

$$a_{k+1}(x,t) = \int_{(t+x)/2}^\infty \Big(\int_0^{(t-x)/2} q(\alpha-\beta)a_k(\alpha-\beta,\alpha+\beta)\,d\beta\Big)\,d\alpha.$$

Taking $H_k(\alpha,\beta) = a_k(\alpha-\beta,\alpha+\beta),\ t+x=2u,\ t-x=2v,$ we calculate for $0 \le v \le u,$

$$H_1(u,v) = \frac{1}{2}\int_u^\infty q(s)\,ds,\ H_{k+1}(u,v) = \int_u^\infty \Big(\int_0^v q(\alpha-\beta)H_k(\alpha,\beta)\,d\beta\Big)\,d\alpha. \qquad (2.7.69)$$

It can be shown by induction that

$$|H_{k+1}(u,v)| \le \frac{1}{2}Q_0(u)\frac{(Q_1(u-v)-Q_1(u))^k}{k!}, \quad k \ge 0,\ 0 \le v \le u. \qquad (2.7.70)$$

Indeed, for $k=0,$ (2.7.70) is obvious. Suppose that (2.7.70) is valid for $H_k(u,v)$. Then (2.7.69) implies

$$|H_{k+1}(u,v)| \le \frac{1}{2}\int_u^\infty Q_0(\alpha)\Big(\int_0^v |q(\alpha-\beta)|\frac{(Q_1(\alpha-\beta)-Q_1(\alpha))^{k-1}}{(k-1)!}\,d\beta\Big)\,d\alpha.$$

Since the functions $Q_0(x)$ and $Q_1(x)$ are monotonic, we get

$$|H_{k+1}(u,v)| \le \frac{1}{2}\cdot\frac{Q_0(u)}{(k-1)!}\int_u^\infty (Q_1(\alpha-v)-Q_1(\alpha))^{k-1}(Q_0(\alpha-v)-Q_0(\alpha))\,d\alpha$$

$$= \frac{1}{2}Q_0(u)\frac{(Q_1(u-v)-Q_1(u))^k}{k!},$$

i.e. (2.7.70) is proved. Therefore, the series

$$H(u,v) = \sum_{k=1}^\infty H_k(u,v)$$

converges absolutely and uniformly for $0 \le v \le u,$ and

$$H(u,v) = \frac{1}{2}\int_u^\infty q(s)\,ds + \int_u^\infty \Big(\int_0^v q(\alpha-\beta)H(\alpha,\beta)\,d\beta\Big)\,d\alpha, \qquad (2.7.71)$$

$$|H(u,v)| \le \frac{1}{2}Q_0(u)\exp\Big(Q_1(u-v)-Q_1(u)\Big). \qquad (2.7.72)$$

Put

$$A(x,t) = H\Big(\frac{t+x}{2},\frac{t-x}{2}\Big). \qquad (2.7.73)$$

Then

$$A(x,t) = \sum_{k=1}^\infty a_k(x,t),$$

the series converges absolutely and uniformly for $0 \le x \le t$, and (2.7.62)-(2.7.64) are valid. Using (2.7.64) we calculate

$$\int_x^\infty |A(x,t)|\,dt \le \exp(Q_1(x)) \int_x^\infty Q_0(\xi)\exp(-Q_1(\xi))\,d\xi$$

$$= \exp(Q_1(x)) \int_x^\infty \frac{d}{d\xi}\Big(\exp(-Q_1(\xi))\Big)\,d\xi = \exp(Q_1(x)) - 1,$$

and we arrive at (2.7.65).

Furthermore, it follows from (2.7.71) that

$$\frac{\partial H(u,v)}{\partial u} = -\frac{1}{2}q(u) - \int_0^v q(u-\beta)H(u,\beta)\,d\beta, \tag{2.7.74}$$

$$\frac{\partial H(u,v)}{\partial v} = \int_u^\infty q(\alpha - v)H(\alpha,v)\,d\alpha. \tag{2.7.75}$$

It follows from (2.7.74)-(2.7.75) and (2.7.72) that

$$\Big|\frac{\partial H(u,v)}{\partial u} + \frac{1}{2}q(u)\Big| \le \frac{1}{2}\int_0^v |q(u-\beta)|Q_0(u)\exp(Q_1(u-\beta) - Q_1(u))\,d\beta,$$

$$\Big|\frac{\partial H(u,v)}{\partial v}\Big| \le \frac{1}{2}\int_u^\infty |q(\alpha - v)|Q_0(\alpha)\exp(Q_1(\alpha - v) - Q_1(\alpha))\,d\alpha.$$

Since

$$\int_0^v |q(u-\beta)|\,d\beta = \int_{u-v}^u |q(s)|\,ds \le Q_0(u-v),$$

$$Q_1(\alpha - v) - Q_1(\alpha) = \int_{\alpha - v}^\alpha Q_0(t)\,dt$$

$$\le \int_{u-v}^u Q_0(t)\,dt = Q_1(u-v) - Q_1(u), \quad u \le \alpha,$$

we get

$$\Big|\frac{\partial H(u,v)}{\partial u} + \frac{1}{2}q(u)\Big| \le \frac{1}{2}Q_0(u)\exp(Q_1(u-v) - Q_1(u))\int_0^v |q(u-\beta)|\,d\beta$$

$$\le \frac{1}{2}Q_0(u-v)Q_0(u)\exp(Q_1(u-v) - Q_1(u)), \tag{2.7.76}$$

$$\Big|\frac{\partial H(u,v)}{\partial v}\Big| \le \frac{1}{2}Q_0(u)\exp(Q_1(u-v) - Q_1(u))\int_u^\infty |q(\alpha - v)|\,d\alpha$$

$$\le \frac{1}{2}Q_0(u-v)Q_0(u)\exp(Q_1(u-v) - Q_1(u)). \tag{2.7.77}$$

By virtue of (2.7.73),

$$\frac{\partial A(x,t)}{\partial x} = \frac{1}{2}\Big(\frac{\partial H(u,v)}{\partial u} - \frac{\partial H(u,v)}{\partial v}\Big),$$

$$\frac{\partial A(x,t)}{\partial t} = \frac{1}{2}\Big(\frac{\partial H(u,v)}{\partial u} + \frac{\partial H(u,v)}{\partial v}\Big),$$

where

$$u=\frac{t+x}{2},\quad v=\frac{t-x}{2}.$$

Hence,

$$\frac{\partial A(x,t)}{\partial x}+\frac{1}{4}q\Big(\frac{x+t}{2}\Big)=\frac{1}{2}\Big(\frac{\partial H(u,v)}{\partial u}+\frac{1}{2}q(u)\Big)-\frac{1}{2}\frac{\partial H(u,v)}{\partial v},$$
$$\frac{\partial A(x,t)}{\partial t}+\frac{1}{4}q\Big(\frac{x+t}{2}\Big)=\frac{1}{2}\Big(\frac{\partial H(u,v)}{\partial u}+\frac{1}{2}q(u)\Big)+\frac{1}{2}\frac{\partial H(u,v)}{\partial v}.$$

Taking (2.7.76) and (2.7.77) into account we arrive at (2.7.66). □

Theorem 2.7.10. *For each* $\delta>0$, *there exists* $a=a_\delta\geq 0$ *such that equation* (2.7.30) *has a unique solution* $y=E(x,\rho)$, $\rho\in\Omega_\delta$, *satisfying the integral equation*

$$E(x,\rho)=\exp(-i\rho x)+\frac{1}{2i\rho}\int_a^x\exp(i\rho(x-t))q(t)E(t,\rho)\,dt$$
$$+\frac{1}{2i\rho}\int_x^\infty\exp(i\rho(t-x))q(t)E(t,\rho)\,dt. \tag{2.7.78}$$

The function $E(x,\rho)$, *called the Birkhoff solution for* (2.7.30)*, has the following properties:*
(i_1) $E^{(\nu)}(x,\rho)=(-i\rho)^\nu\exp(-i\rho x)(1+o(1))$, $x\to\infty$ $\nu=0,1$, *uniformly for* $|\rho|\geq\delta$, $Im\,\rho\geq\alpha$, *for each fixed* $\alpha>0$;
(i_2) $E^{(\nu)}(x,\rho)=(-i\rho)^\nu\exp(-i\rho x)(1+O(\rho^{-1}))$, $|\rho|\to\infty$ $\rho\in\Omega$, *uniformly for* $x\geq a$;
(i_3) *for each fixed* $x\geq 0$, *the functions* $E^{(\nu)}(x,\rho)$ *are analytic for* $Im\,\rho>0$, $|\rho|\geq\delta$, *and are continuous for* $\rho\in\Omega_\delta$;
(i_4) *the functions* $e(x,\rho)$ *and* $E(x,\rho)$ *form a fundamental system of solutions for* (2.7.30)*, and* $\langle e(x,\rho),E(x,\rho)\rangle=-2i\rho$.
(i_5) *If* $\delta\geq Q_0(0)$, *then one can take above* $a=0$.

Proof. For fixed $\delta>0$ choose $a=a_\delta\geq 0$ such that $Q_0(a)\leq\delta$. We transform (2.7.78) by means of the replacement $E(x,\rho)=\exp(-i\rho x)\xi(x,\rho)$ to the equation

$$\xi(x,\rho)=1+\frac{1}{2i\rho}\int_a^x\exp(2i\rho(x-t))q(t)\xi(t,\rho)\,dt$$
$$+\frac{1}{2i\rho}\int_x^\infty q(t)\xi(t,\rho)\,dt. \tag{2.7.79}$$

The method of successive approximations gives

$$\xi_0(x,\rho)=1,$$
$$\xi_{k+1}(x,\rho)=\frac{1}{2i\rho}\int_a^x\exp(2i\rho(x-t))q(t)\xi_k(t,\rho)\,dt+\frac{1}{2i\rho}\int_x^\infty q(t)\xi_k(t,\rho)\,dt,$$
$$\xi(x,\rho)=\sum_{k=0}^\infty\xi_k(x,\rho).$$

This yields

$$|\xi_{k+1}(x,\rho)|\leq\frac{1}{2|\rho|}\int_a^\infty|q(t)\xi_k(t,\rho)|\,dt,$$

and hence

$$|\xi_k(x,\rho)| \le \Big(\frac{(Q_0(a)}{2|\rho|}\Big)^k.$$

Thus for $x \ge a, |\rho| \ge Q_0(a),$ we get

$$|\xi(x,\rho)| \le 2, \quad |\xi(x,\rho) - 1| \le \frac{Q_0(a)}{|\rho|}.$$

It follows from (2.7.78) that

$$E'(x,\rho) = \exp(-i\rho x)$$
$$\times\Big(-i\rho + \frac{1}{2}\int_a^x \exp(2i\rho(x-t))q(t)\xi(t,\rho)\,dt - \frac{1}{2}\int_x^\infty q(t)\xi(t,\rho)\,dt\Big). \qquad (2.7.80)$$

Since $|\xi(x,\rho)| \le 2$ for $x \ge a,\ \rho \in \Omega_\delta,$ it follows from (2.7.79) and (2.7.80) that

$$|E^{(\nu)}(x,\rho)(-i\rho)^{-\nu}\exp(i\rho x) - 1|$$
$$\le \frac{1}{|\rho|}\Big(\int_a^x \exp(-2\tau(x-t))|q(t)|\,dt + \int_x^\infty |q(t)|\,dt\Big)$$
$$\le \frac{1}{|\rho|}\Big(\exp(-\tau x)\int_a^{x/2} |q(t)|\,dt + \int_{x/2}^\infty |q(t)|\,dt\Big),$$

and consequently $(i_1) - (i_2)$ are proved.
The other assertions of Theorem 2.7.10 are obvious. □

Properties of the spectrum. Denote

$$\Delta(\rho) = e'(0,\rho) - h\,e(0,\rho). \qquad (2.7.81)$$

By virtue of Theorem 2.7.7, the function $\Delta(\rho)$ is analytic for $Im\,\rho > 0,$ and continuous for $\rho \in \Omega.$ It follows from (2.7.34) that for $|\rho| \to \infty,\ \rho \in \Omega,$

$$e(0,\rho) = 1 + \frac{\omega_1}{i\rho} + o\Big(\frac{1}{\rho}\Big), \quad \Delta(\rho) = (i\rho)\Big(1 + \frac{\omega_{11}}{i\rho} + o\Big(\frac{1}{\rho}\Big)\Big), \qquad (2.7.82)$$

where $\omega_1 = \omega(0),\ \omega_{11} = \omega(0) - h.$ Using (2.7.36), (2.7.45) and (2.7.49) one can obtain more precisely

$$\left.\begin{aligned} e(0,\rho) &= 1 + \frac{\omega_1}{i\rho} + \frac{1}{2i\rho}\int_0^\infty q(t)\exp(2i\rho t)\,dt + O\Big(\frac{1}{\rho^2}\Big), \\ \Delta(\rho) &= (i\rho)\Big(1 + \frac{\omega_{11}}{i\rho} - \frac{1}{2i\rho}\int_0^\infty q(t)\exp(2i\rho t)\,dt + O\Big(\frac{1}{\rho^2}\Big)\Big). \end{aligned}\right\} \qquad (2.7.83)$$

Denote

$$\Lambda = \{\lambda = \rho^2 : \rho \in \Omega,\ \Delta(\rho) = 0\},$$
$$\Lambda' = \{\lambda = \rho^2 :\ Im\,\rho > 0, \Delta(\rho) = 0\},$$
$$\Lambda'' = \{\lambda = \rho^2 :\ Im\,\rho = 0, \rho \ne 0, \Delta(\rho) = 0\}.$$

Obviously, $\Lambda = \Lambda' \cup \Lambda''$ is a bounded set, and Λ' is a bounded and at most countable set.

Denote

$$\Phi(x,\lambda) = \frac{e(x,\rho)}{\Delta(\rho)}. \tag{2.7.84}$$

The function $\Phi(x,\lambda)$ satisfies (2.7.30) and on account of (2.7.81) and Theorem 2.7.7 also the conditions

$$U(\Phi) = 1, \tag{2.7.85}$$

$$\Phi(x,\lambda) = O(\exp(i\rho x)), \quad x \to \infty, \rho \in \Omega, \tag{2.7.86}$$

where U is defined by (2.7.31). The function $\Phi(x,\lambda)$ is called the *Weyl solution* for L. Note that (2.7.30), (2.7.85) and (2.7.86) uniquely determine the Weyl solution.

Denote $M(\lambda) := \Phi(0,\lambda)$. The function $M(\lambda)$ is called the *Weyl function* for L. It follows from (2.7.84) that

$$M(\lambda) = \frac{e(0,\rho)}{\Delta(\rho)}. \tag{2.7.87}$$

Clearly,

$$\Phi(x,\lambda) = S(x,\lambda) + M(\lambda)\varphi(x,\lambda), \tag{2.7.88}$$

where the functions $\varphi(x,\lambda)$ and $S(x,\lambda)$ are solutions of (2.7.30) under the initial conditions

$$\varphi(0,\lambda) = 1, \varphi'(0,\lambda) = h, \quad S(0,\lambda) = 0, S'(0,\lambda) = 1.$$

We recall that the Weyl function plays an important role in the spectral theory of Sturm-Liouville operators (see [19] for more details).

By virtue of Liouville‘s formula for the Wronskian [13, p.83], $\langle \varphi(x,\lambda), \Phi(x,\lambda) \rangle$ does not depend on x. Since for $x = 0$,

$$\langle \varphi(x,\lambda), \Phi(x,\lambda) \rangle_{|x=0} = U(\Phi) = 1,$$

we infer

$$\langle \varphi(x,\lambda), \Phi(x,\lambda) \rangle \equiv 1. \tag{2.7.89}$$

Theorem 2.7.11. *The Weyl function* $M(\lambda)$ *is analytic in* $\Pi \setminus \Lambda'$ *and continuous in* $\Pi_1 \setminus \Lambda$. *The set of singularities of* $M(\lambda)$ (*as an analytic function*) *coincides with the set* $\Lambda_0 := \{\lambda : \lambda \geq 0\} \cup \Lambda$.

Theorem 2.7.11 follows from (2.7.81), (2.7.87) and Theorem 2.7.7. By virtue of (2.7.88), the set of singularities of the Weyl solution $\Phi(x,\lambda)$ coincides with Λ_0 for all $x \geq 0$, since the functions $\varphi(x,\lambda)$ and $S(x,\lambda)$ are entire in λ for each fixed $x \geq 0$.

Definition 2.7.1. The set of singularities of the Weyl function $M(\lambda)$ is called the spectrum of L. The values of the parameter λ, for which equation (2.7.30) has nontrivial solutions satisfying the conditions $U(y) = 0, y(\infty) = 0$ (i.e. $\lim_{x\to\infty} y(x) = 0$), are called eigenvalues of L, and the corresponding solutions are called eigenfunctions.

Remark 2.7.5. One can introduce the operator

$$L^o : D(L^o) \to L_2(0,\infty), \ y \to -y'' + q(x)y$$

with the domain of definition $D(L^o) = \{y :\ y \in L_2(I) \cap AC_{loc}(I), y' \in AC_{loc}(I)\,,\ L^o y \in L_2(I), U(y) = 0\}$, where $I := [0,\infty)$. It is easy to verify that the spectrum of L^o coincides with Λ_0. For the Sturm-Liouville equation there is no difference between working either with the operator L^o or with the pair L. However, for generalizations for many other classes of inverse problems, from methodical point of view it is more natural to consider the pair L (see, for example, [19]).

Theorem 2.7.12. *L has no eigenvalues $\lambda > 0$.*

Proof. Suppose that $\lambda_0 = \rho_0^2 > 0$ is an eigenvalue, and let $y_0(x)$ be a corresponding eigenfunction. Since the functions $\{e(x,\rho_0), e(x,-\rho_0)\}$ form a fundamental system of solutions of (2.7.30), we have $y_0(x) = Ae(x,\rho_0) + Be(x,-\rho_0)$. For $x \to \infty$, $y_0(x) \sim 0$, $e(x,\pm\rho_0) \sim \exp(\pm i\rho_0 x)$. But this is possible only if $A = B = 0$. □

Theorem 2.7.13. *If $(1+x)q(x) \in L(0,\infty)$, then $\lambda = 0$ is not an eigenvalue of L.*

Proof. The function $e(x) := e(x,0)$ is a solution of (2.7.30) for $\lambda = 0$, and according to Theorem 2.7.8,

$$\lim_{x\to\infty} e(x) = 1.$$

Take $a > 0$ such that

$$e(x) \ge \frac{1}{2} \ \text{ for } x \ge a,$$

and consider the function

$$z(x) := e(x)\int_a^x \frac{dt}{e^2(t)}.$$

It is easy to check that

$$z''(x) = q(x)z(x), \quad \lim_{x\to\infty} z(x) = +\infty,$$

and

$$e(x)z'(x) - e'(x)z(x) \equiv 1.$$

Suppose that $\lambda = 0$ is an eigenvalue, and let $y_0(x)$ be a corresponding eigenfunction. Since the functions $\{e(x), z(x)\}$ form a fundamental system of solutions of (2.7.30) for $\lambda = 0$, we have

$$y_0(x) = C_1^0 e(x) + C_2^0 z(x).$$

It follows from above that this is possible only if $C_1^0 = C_2^0 = 0$. □

Remark 2.7.6. Let

$$q(x) = \frac{2a^2}{(1+ax)^2}, \quad h = -a,$$

where a is a complex number such that $a \notin (-\infty, 0]$. Then $q(x) \in L(0,\infty)$, but $xq(x) \notin L(0,\infty)$. In this case $\lambda = 0$ is an eigenvalue, and by differentiation one verifies that

$$y(x) = \frac{1}{1+ax}$$

is the corresponding eigenfunction.

Theorem 2.7.14. Let $\lambda_0 \notin [0,\infty)$. For λ_0 to be an eigenvalue, it is necessary and sufficient that $\Delta(\rho_0) = 0$. In other words, the set of nonzero eigenvalues coincides with

Λ'. For each eigenvalue $\lambda_0 \in \Lambda'$ there exists only one (up to a multiplicative constant) eigenfunction, namely,

$$\varphi(x,\lambda_0) = \beta_0 e(x,\rho_0), \quad \beta_0 \neq 0. \tag{2.7.90}$$

Proof. Let $\lambda_0 \in \Lambda'$. Then $U(e(x,\rho_0)) = \Delta(\rho_0) = 0$ and, by virtue of (2.7.33), $\lim_{x\to\infty} e(x,\rho_0) = 0$. Thus, $e(x,\rho_0)$ is an eigenfunction, and $\lambda_0 = \rho_0^2$ is an eigenvalue. Moreover, it follows from (2.7.84) and (2.7.89) that $\langle \varphi(x,\lambda), e(x,\rho)\rangle = \Delta(\rho)$, and consequently (2.7.90) is valid.

Conversely, let $\lambda_0 = \rho_0^2, Im\rho_0 > 0$ be an eigenvalue, and let $y_0(x)$ be a corresponding eigenfunction. Clearly, $y_0(0) \neq 0$. Without loss of generality we put $y_0(0) = 1$. Then $y_0'(0) = h$, and hence $y_0(x) \equiv \varphi(x,\lambda_0)$. Since the functions $E(x,\rho_0)$ and $e(x,\rho_0)$ form a fundamental system of solutions of equation (2.7.30), we get $y_0(x) = \alpha_0 E(x,\rho_0) + \beta_0 e(x,\rho_0)$. As $x \to \infty$, we calculate $\alpha_0 = 0$, i.e. $y_0(x) = \beta_0 e(x,\rho_0)$. This yields (2.7.90). Consequently, $\Delta(\rho_0) = U(e(x,\rho_0)) = 0$, and $\varphi(x,\lambda_0)$ and $e(x,\rho_0)$ are eigenfunctions. □

Thus, the spectrum of L consists of the positive half-line $\{\lambda : \lambda \geq 0\}$, and the discrete set $\Lambda = \Lambda' \cup \Lambda''$. Each element of Λ' is an eigenvalue of L. According to Theorem 2.7.12, the points of Λ'' are not eigenvalues of L, they are called *spectral singularities* of L.

Example 2.7.2. Let $q(x) \equiv 0$, $h = i\theta$, where θ is a real number. Then $\Delta(\rho) = i\rho - h$, and $\Lambda' = \emptyset$, $\Lambda'' = \{\theta\}$, i.e. L has no eigenvalues, and the point $\rho_0 = \theta$ is a spectral singularity for L.

It follows from (2.7.34), (2.7.82), (2.7.84) and (2.7.87) that for $|\rho| \to \infty$, $\rho \in \Omega$,

$$M(\lambda) = \frac{1}{i\rho}\left(1 + \frac{m_1}{i\rho} + o\left(\frac{1}{\rho}\right)\right), \tag{2.7.91}$$

$$\Phi^{(\nu)}(x,\lambda) = (i\rho)^{\nu-1}\exp(i\rho x)\left(1 + \frac{B(x)}{i\rho} + o\left(\frac{1}{\rho}\right)\right), \tag{2.7.92}$$

uniformly for $x \geq 0$; here

$$m_1 = h, \quad B(x) = h + \frac{1}{2}\int_0^x q(s)\,ds.$$

Taking (2.7.83) into account one can derive more precisely

$$M(\lambda) = \frac{1}{i\rho}\left(1 + \frac{m_1}{i\rho} + \frac{1}{i\rho}\int_0^\infty q(t)\exp(2i\rho t)\,dt + O\left(\frac{1}{\rho^2}\right)\right), \tag{2.7.93}$$

$$|\rho| \to \infty, \quad \rho \in \Omega.$$

Moreover, if $q(x) \in W_N$, then by virtue of (2.7.50) we get

$$M(\lambda) = \frac{1}{i\rho}\left(1 + \sum_{s=1}^{N+1}\frac{m_s}{(i\rho)^s} + o\left(\frac{1}{\rho^{N+1}}\right)\right), \quad |\rho| \to \infty,\ \rho \in \Omega, \tag{2.7.94}$$

where $m_1 = h, \quad m_2 = -\frac{1}{2}q(0) + h^2.$

Denote

$$V(\lambda) = \frac{1}{2\pi i}\left(M^-(\lambda) - M^+(\lambda)\right), \quad \lambda > 0, \tag{2.7.95}$$

where

$$M^{\pm}(\lambda) = \lim_{z\to 0, Re z>0} M(\lambda \pm iz).$$

It follows from (2.7.91) and (2.7.95) that for $\rho > 0,\ \rho \to +\infty$,

$$V(\lambda) = \frac{1}{2\pi i}\Big(-\frac{1}{i\rho}\Big(1 - \frac{m_1}{i\rho} + o\Big(\frac{1}{\rho}\Big)\Big) - \frac{1}{i\rho}\Big(1 + \frac{m_1}{i\rho} + o\Big(\frac{1}{\rho}\Big)\Big)\Big),$$

and consequently

$$V(\lambda) = \frac{1}{\pi\rho}\left(1 + o\left(\frac{1}{\rho}\right)\right), \quad \rho > 0,\ \rho \to +\infty. \tag{2.7.96}$$

In view of (2.7.93), we calculate more precisely

$$V(\lambda) = \frac{1}{\pi\rho}\left(1 + \frac{1}{\rho}\int_0^\infty q(t)\sin 2\rho t\, dt + O\left(\frac{1}{\rho^2}\right)\right),\ \rho > 0,\ \rho \to +\infty. \tag{2.7.97}$$

Moreover, if $q(x) \in W_{N+1}$, then (2.7.94) implies

$$V(\lambda) = \frac{1}{\pi\rho}\left(1 + \sum_{s=1}^{N+1}\frac{V_s}{\rho^s} + o\left(\frac{1}{\rho^{N+1}}\right)\right), \quad \rho > 0,\ \rho \to +\infty, \tag{2.7.98}$$

where $V_{2s} = (-1)^s m_{2s},\ V_{2s+1} = 0$.

An expansion theorem. In the λ - plane we consider the contour $\gamma = \gamma' \cup \gamma''$ (with counterclockwise circuit), where γ' is a bounded closed contour encircling the set $\Lambda \cup \{0\}$, and γ'' is the two-sided cut along the arc $\{\lambda:\ \lambda > 0,\ \lambda \notin int\,\gamma'\}$.

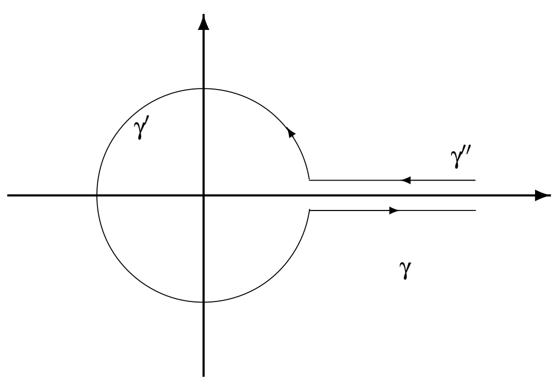

Figure 2.7.1.

Theorem 2.7.15. *Let* $f(x) \in W_2$. *Then, uniformly for* $x \geq 0$,

$$f(x) = \frac{1}{2\pi i} \int_{\gamma} \varphi(x,\lambda) F(\lambda) M(\lambda)\, d\lambda, \tag{2.7.99}$$

where

$$F(\lambda) := \int_0^{\infty} \varphi(t,\lambda) f(t)\, dt.$$

Proof. We will use the contour integral method. For this purpose we consider the function

$$Y(x,\lambda) = \Phi(x,\lambda) \int_0^x \varphi(t,\lambda) f(t)\, dt + \varphi(x,\lambda) \int_x^{\infty} \Phi(t,\lambda) f(t)\, dt. \tag{2.7.100}$$

Since the functions $\varphi(x,\lambda)$ and $\Phi(x,\lambda)$ satisfy (2.7.30) we transform $Y(x,\lambda)$ as follows

$$Y(x,\lambda) = \frac{1}{\lambda} \Phi(x,\lambda) \int_0^x (-\varphi''(t,\lambda) + q(t)\varphi(t,\lambda)) f(t)\, dt$$

$$+ \frac{1}{\lambda} \varphi(x,\lambda) \int_x^{\infty} (-\Phi''(t,\lambda) + q(t)\Phi(t,\lambda)) f(t)\, dt.$$

Two-fold integration by parts of terms with second derivatives yields in view of (2.7.89)

$$Y(x,\lambda) = \frac{1}{\lambda} \Big(f(x) + Z(x,\lambda) \Big), \tag{2.7.101}$$

where

$$Z(x,\lambda) = (f'(0) - hf(0))\Phi(x,\lambda) + \Phi(x,\lambda) \int_0^x \varphi(t,\lambda) \ell f(t)\, dt$$

$$+ \varphi(x,\lambda) \int_x^{\infty} \Phi(t,\lambda) \ell f(t)\, dt. \tag{2.7.102}$$

Similarly,

$$F(\lambda) = -\frac{1}{\lambda} \Big(f'(0) - hf(0) \Big) + \frac{1}{\lambda} \int_0^{\infty} \varphi(t,\lambda) \ell f(t)\, dt, \quad \lambda > 0. \tag{2.7.103}$$

The function $\varphi(x,\lambda)$ satisfies the integral equation (2.2.47). Denote

$$\mu_T(\lambda) = \max_{0 \leq x \leq T} (|\varphi(x,\lambda)| \exp(-|\tau| x)), \quad \tau := Im\,\rho.$$

Then (2.2.47) gives for $|\rho| \geq 1$, $x \in [0,T]$,

$$|\varphi(x,\lambda)| \exp(-|\tau| x) \leq C + \frac{\mu_T(\lambda)}{|\rho|} \int_0^T |q(t)|\, dt,$$

and consequently

$$\mu_T(\lambda) \leq C + \frac{\mu_T(\lambda)}{|\rho|} \int_0^T |q(t)|\, dt \leq C + \frac{\mu_T(\lambda)}{|\rho|} \int_0^{\infty} |q(t)|\, dt.$$

From this we get $|\mu_T(\lambda)| \le C$ for $|\rho| \ge \rho^*$. Together with (2.2.48) this yields for $\nu = 0,1$, $|\rho| \ge \rho^*$,

$$|\varphi^{(\nu)}(x,\lambda)| \le C|\rho|^{\nu} \exp(|\tau| x), \tag{2.7.104}$$

uniformly for $x \ge 0$. Furthermore, it follows from (2.7.92) that for $\nu = 0,1$, $|\rho| \ge \rho^*$,

$$|\Phi^{(\nu)}(x,\lambda)| \le C|\rho|^{\nu-1} \exp(-|\tau| x), \tag{2.7.105}$$

uniformly for $x \ge 0$. By virtue of (2.7.102), (2.7.104) and (2.7.105) we get

$$Z(x,\lambda) = O\Big(\frac{1}{\rho}\Big), \quad |\lambda| \to \infty,$$

uniformly for $x \ge 0$. Hence, (2.7.101) implies

$$\lim_{R\to\infty} \sup_{x\ge 0} \left| f(x) - \frac{1}{2\pi i} \int_{|\lambda|=R} Y(x,\lambda)\, d\lambda \right| = 0, \tag{2.7.106}$$

where the contour in the integral is used with counterclockwise circuit. Consider the contour $\gamma_R^0 = (\gamma \cap \{\lambda : |\lambda| \le R\}) \cup \{\lambda : |\lambda| = R\}$ (with clockwise circuit).

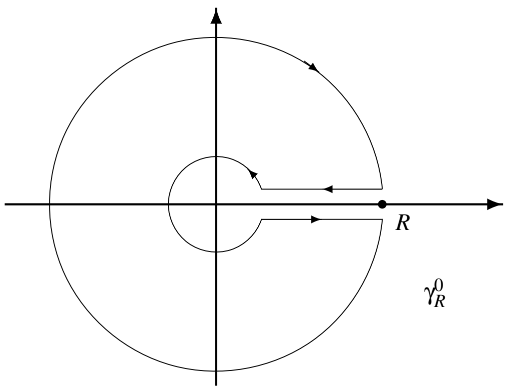

Figure 2.7.2.

By Cauchy‘s theorem [14, p.85]

$$\frac{1}{2\pi i} \int_{\gamma_R^0} Y(x,\lambda) d\lambda = 0.$$

Taking (2.7.106) into account we obtain

$$\lim_{R\to\infty} \sup_{x\ge 0} \left| f(x) - \frac{1}{2\pi i} \int_{\gamma_R} Y(x,\lambda)\, d\lambda \right| = 0,$$

where $\gamma_R = \gamma \cap \{\lambda : |\lambda| \le R\}$ (with counterclockwise circuit). From this, using (2.7.100) and (2.7.88), we arrive at (2.7.99), since the terms with $S(x,\lambda)$ vanish by Cauchy‘s theorem. We note that according to (2.7.97), (2.7.103) and (2.7.104),

$$F(\lambda) = O\Big(\frac{1}{\lambda}\Big),\ M(\lambda) = O\Big(\frac{1}{\rho}\Big),\ \varphi(x,\lambda) = O(1), \quad x \ge 0,\ \lambda > 0,\ \lambda \to \infty,$$

and consequently the integral in (2.7.99) converges absolutely and uniformly for $x \ge 0$. Theorem 2.7.15 is proved. □

Remark 2.7.7. If $q(x)$ and h are real, and $(1+x)q(x) \in L(0,\infty)$, then (see [27, Sec. 2.3]) $\Lambda'' = \emptyset$, $\Lambda' \subset (-\infty, 0)$ is a finite set of simple eigenvalues, $V(\lambda) > 0$ for $\lambda > 0$ ($V(\lambda)$ is defined by (2.7.95)), and $M(\lambda) = O(\rho^{-1})$ as $\rho \to 0$. Then (2.7.99) takes the form

$$f(x) = \int_0^\infty \varphi(x,\lambda)F(\lambda)V(\lambda)d\lambda + \sum_{\lambda_j \in \Lambda'} \varphi(x,\lambda_j)F(\lambda_j)Q_j, \quad Q_j := \operatorname*{Res}_{\lambda=\lambda_j} M(\lambda).$$

or

$$f(x) = \int_{-\infty}^{\infty} \varphi(x,\lambda)F(\lambda)\, d\sigma(\lambda),$$

where $\sigma(\lambda)$ is the spectral function of L (see [19]). For $\lambda < 0$, $\sigma(\lambda)$ is a step-function; for $\lambda > 0$, $\sigma(\lambda)$ is an absolutely continuous function, and $\sigma'(\lambda) = V(\lambda)$.

Remark 2.7.8. It follows from the proof that Theorem 2.7.15 remains valid also for $f(x) \in W_1$.

4. Recovery of the differential equation from the Weyl function

In this subsection we study the inverse problem of recovering the pair $L = L(q(x), h)$ of the form (2.7.30)-(2.7.31) from the given Weyl function $M(\lambda)$. For this purpose we will use the method of spectral mappings (see [17], [18]). First, let us prove the uniqueness theorem for the solution of the inverse problem.

Theorem 2.7.16. *If $M(\lambda) = \tilde{M}(\lambda)$, then $L = \tilde{L}$. Thus, the specification of the Weyl function uniquely determines $q(x)$ and h.*

Proof. Let us define the matrix $P(x,\lambda) = [P_{jk}(x,\lambda)]_{j,k=1,2}$ by the formula

$$P(x,\lambda) \begin{bmatrix} \tilde{\varphi}(x,\lambda) & \tilde{\Phi}(x,\lambda) \\ \tilde{\varphi}'(x,\lambda) & \tilde{\Phi}'(x,\lambda) \end{bmatrix} = \begin{bmatrix} \varphi(x,\lambda) & \Phi(x,\lambda) \\ \varphi'(x,\lambda) & \Phi'(x,\lambda) \end{bmatrix}.$$

By virtue of (2.7.89), this yields

$$\left.\begin{aligned} P_{j1}(x,\lambda) &= \varphi^{(j-1)}(x,\lambda)\tilde{\Phi}'(x,\lambda) - \Phi^{(j-1)}(x,\lambda)\tilde{\varphi}'(x,\lambda) \\ P_{j2}(x,\lambda) &= \Phi^{(j-1)}(x,\lambda)\tilde{\varphi}(x,\lambda) - \varphi^{(j-1)}(x,\lambda)\tilde{\Phi}(x,\lambda) \end{aligned}\right\}, \tag{2.7.107}$$

$$\left.\begin{aligned} \varphi(x,\lambda) &= P_{11}(x,\lambda)\tilde{\varphi}(x,\lambda) + P_{12}(x,\lambda)\tilde{\varphi}'(x,\lambda) \\ \Phi(x,\lambda) &= P_{11}(x,\lambda)\tilde{\Phi}(x,\lambda) + P_{12}(x,\lambda)\tilde{\Phi}'(x,\lambda) \end{aligned}\right\}. \tag{2.7.108}$$

Using (2.7.107), (2.7.104)-(2.7.105) we get for $|\lambda| \to \infty$, $\lambda = \rho^2$,

$$P_{jk}(x,\lambda) - \delta_{jk} = O(\rho^{-1}),\ j \le k; \quad P_{21}(x,\lambda) = O(1). \tag{2.7.109}$$

If $M(\lambda) \equiv \tilde{M}(\lambda)$, then in view of (2.7.107) and (2.7.88), for each fixed x, the functions $P_{jk}(x,\lambda)$ are entire in λ. Together with (2.7.109) this yields $P_{11}(x,\lambda) \equiv 1$, $P_{12}(x,\lambda) \equiv 0$.

Substituting into (2.7.108) we get $\varphi(x,\lambda) \equiv \tilde{\varphi}(x,\lambda)$, $\Phi(x,\lambda) \equiv \tilde{\Phi}(x,\lambda)$ for all x and λ, and consequently, $L = \tilde{L}$. □

Let us now start to construct the solution of the inverse problem. We shall say that $L \in V_N$ if $q(x) \in W_N$. We shall subsequently solve the inverse problem in the classes V_N.

Let a model pair $\tilde{L} = L(\tilde{q}(x), \tilde{h})$ be chosen such that

$$\int_{\rho^*}^{\infty} \rho^4 |\hat{V}(\lambda)|^2 \, d\rho < \infty, \quad \hat{V} := V - \tilde{V} \tag{2.7.110}$$

for sufficiently large $\rho^* > 0$. The condition (2.7.110) is needed for technical reasons. In principal, one could take any $\tilde{L}$ (for example, with $\tilde{q}(x) = \tilde{h} = 0$) but generally speaking proofs would become more complicated. On the other hand, (2.7.110) is not a very strong restriction, since by virtue of (2.7.97),

$$\rho^2 \hat{V}(\lambda) = \frac{1}{\pi} \int_0^{\infty} \hat{q}(t) \sin 2\rho t \, dt + O\Big(\frac{1}{\rho}\Big).$$

In particular, if $q(x) \in L_2$, then (2.7.110) is fulfilled automatically for any $\tilde{q}(x) \in L_2$. Hence, for $N \geq 1$, the condition (2.7.110) is fulfilled for any model $\tilde{L} \in V_N$.

It follows from (2.7.110) that

$$\int_{\lambda^*}^{\infty} |\hat{V}(\lambda)| \, d\lambda = 2 \int_{\rho^*}^{\infty} \rho |\hat{V}(\lambda)| \, d\rho < \infty, \quad \lambda^* = (\rho^*)^2, \ \lambda = \rho^2. \tag{2.7.111}$$

Denote

$$\left.\begin{aligned} D(x,\lambda,\mu) &= \frac{\langle \varphi(x,\lambda), \varphi(x,\mu) \rangle}{\lambda - \mu} = \int_0^x \varphi(t,\lambda)\varphi(t,\mu)\, dt, \\ \tilde{D}(x,\lambda,\mu) &= \frac{\langle \tilde{\varphi}(x,\lambda), \tilde{\varphi}(x,\mu) \rangle}{\lambda - \mu} = \int_0^x \tilde{\varphi}(t,\lambda)\tilde{\varphi}(t,\mu)\, dt, \\ r(x,\lambda,\mu) &= D(x,\lambda,\mu)\hat{M}(\mu), \quad \tilde{r}(x,\lambda,\mu) = \tilde{D}(x,\lambda,\mu)\hat{M}(\mu). \end{aligned}\right\} \tag{2.7.112}$$

Lemma 2.7.7. *The following estimate holds*

$$|D(x,\lambda,\mu)|, \ |\tilde{D}(x,\lambda,\mu)| \leq \frac{C_x \exp(|Im\rho| x)}{|\rho \mp \theta| + 1}, \tag{2.7.113}$$

$$\lambda = \rho^2, \ \mu = \theta^2 \geq 0, \ \pm\theta Re\rho \geq 0.$$

Proof. Let $\rho = \sigma + i\tau$. For definiteness, let $\theta \geq 0$ and $\sigma \geq 0$. All other cases can be treated in the same way. Take a fixed $\delta_0 > 0$. For $|\rho - \theta| \geq \delta_0$ we have by virtue of (2.7.112) and (2.7.104),

$$|D(x,\lambda,\mu)| = \Big| \frac{\langle \varphi(x,\lambda), \varphi(x,\mu) \rangle}{\lambda - \mu} \Big| \leq C \exp(|\tau| x) \frac{|\rho| + |\theta|}{|\rho^2 - \theta^2|}. \tag{2.7.114}$$

Since

$$\frac{|\rho| + |\theta|}{|\rho + \theta|} = \frac{\sqrt{\sigma^2 + \tau^2} + \theta}{\sqrt{(\sigma + \theta)^2 + \tau^2}} \leq \frac{\sqrt{\sigma^2 + \tau^2} + \theta}{\sqrt{\sigma^2 + \tau^2 + \theta^2}} \leq \sqrt{2}$$

(here we use that $(a+b)^2 \le 2(a^2+b^2)$ for all real a,b), (2.7.114) implies

$$|D(x,\lambda,\mu)| \le \frac{C\exp(|\tau|x)}{|\rho-\theta|}. \tag{2.7.115}$$

For $|\rho-\theta| \ge \delta_0$, we get

$$\frac{|\rho-\theta|+1}{|\rho-\theta|} \le 1+\frac{1}{\delta_0},$$

and consequently

$$\frac{1}{|\rho-\theta|} \le \frac{C_0}{|\rho-\theta|+1} \quad \text{with } C_0 = \frac{\delta_0+1}{\delta_0}.$$

Substituting this estimate into the right-hand side of (2.7.115) we obtain

$$|D(x,\lambda,\mu)| \le \frac{C\exp(|\tau|x)}{|\rho-\theta|+1},$$

and (2.7.113) is proved for $|\rho-\theta| \ge \delta_0$.

For $|\rho-\theta| \le \delta_0$, we have by virtue of (2.7.112) and (2.7.104),

$$|D(x,\lambda,\mu)| \le \int_0^x |\varphi(t,\lambda)\varphi(t,\mu)|\,dt \le C_x\exp(|\tau|x),$$

i.e. (2.7.113) is also valid for $|\rho-\theta| \le \delta_0$. □

Lemma 2.7.8. *The following estimates hold*

$$\int_1^\infty \frac{d\theta}{\theta(|R-\theta|+1)} = O\Big(\frac{\ln R}{R}\Big), \quad R\to\infty, \tag{2.7.116}$$

$$\int_1^\infty \frac{d\theta}{\theta^2(|R-\theta|+1)^2} = O\Big(\frac{1}{R^2}\Big), \quad R\to\infty. \tag{2.7.117}$$

Proof. Since

$$\frac{1}{\theta(R-\theta+1)} = \frac{1}{R+1}\Big(\frac{1}{\theta}+\frac{1}{R-\theta+1}\Big),$$

$$\frac{1}{\theta(\theta-R+1)} = \frac{1}{R-1}\Big(\frac{1}{\theta-R+1}-\frac{1}{\theta}\Big),$$

we calculate for $R>1$,

$$\int_1^\infty \frac{d\theta}{\theta(|R-\theta|+1)} = \int_1^R \frac{d\theta}{\theta(R-\theta+1)} + \int_R^\infty \frac{d\theta}{\theta(\theta-R+1)}$$

$$= \frac{1}{R+1}\int_1^R \Big(\frac{1}{\theta}+\frac{1}{R-\theta+1}\Big)d\theta + \frac{1}{R-1}\int_R^\infty \Big(\frac{1}{\theta-R+1}-\frac{1}{\theta}\Big)d\theta$$

$$= \frac{2\ln R}{R+1}+\frac{\ln R}{R-1},$$

i.e. (2.7.116) is valid. Similarly, for $R>1$,

$$\int_1^\infty \frac{d\theta}{\theta^2(|R-\theta|+1)^2} = \int_1^R \frac{d\theta}{\theta^2(R-\theta+1)^2} + \int_R^\infty \frac{d\theta}{\theta^2(\theta-R+1)^2}$$

$$\leq \frac{1}{(R+1)^2}\int_1^R \Big(\frac{1}{\theta}+\frac{1}{R-\theta+1}\Big)^2 d\theta + \frac{1}{R^2}\int_R^\infty \frac{d\theta}{(\theta-R+1)^2}$$

$$= \frac{2}{(R+1)^2}\Big(\int_1^R \frac{d\theta}{\theta^2}+\int_1^R \frac{d\theta}{\theta(R-\theta+1)}\Big)+\frac{1}{R^2}\int_1^\infty \frac{d\theta}{\theta^2} = O\Big(\frac{1}{R^2}\Big),$$

i.e. (2.7.116) is valid. □

In the λ - plane we consider the contour $\gamma = \gamma' \cup \gamma''$ (with counterclockwise circuit), where γ' is a bounded closed contour encircling the set $\Lambda \cup \tilde{\Lambda} \cup \{0\}$, and γ'' is the two-sided cut along the arc $\{\lambda: \lambda > 0, \lambda \notin int\,\gamma'\}$ (see fig. 2.7.1).

Theorem 2.7.17. *The following relations hold*

$$\tilde{\varphi}(x,\lambda) = \varphi(x,\lambda) + \frac{1}{2\pi i}\int_\gamma \tilde{r}(x,\lambda,\mu)\varphi(x,\mu)\,d\mu, \tag{2.7.118}$$

$$r(x,\lambda,\mu) - \tilde{r}(x,\lambda,\mu) + \frac{1}{2\pi i}\int_\gamma \tilde{r}(x,\lambda,\xi) r(x,\xi,\mu)\,d\xi = 0. \tag{2.7.119}$$

Equation (2.7.118) *is called the main equation of the inverse problem.*

Proof. It follows from (2.7.91), (2.7.104), (2.7.112) and (2.7.113) that

$$|r(x,\lambda,\mu)|,\ |\tilde{r}(x,\lambda,\mu)| \leq \frac{C_x}{|\mu|(|\rho \mp \theta|+1)}, \quad |\varphi(x,\lambda)| \leq C, \tag{2.7.120}$$

$$\lambda,\mu \in \gamma, \quad \pm Re\,\rho\, Re\,\theta \geq 0.$$

In view of (2.7.116), it follows from (2.7.120) that the integrals in (2.7.118) and (2.7.119) converge absolutely and uniformly on γ for each fixed $x \geq 0$.

Denote $J_\gamma = \{\lambda: \lambda \notin \gamma \cup int\,\gamma'\}$. Consider the contour $\gamma_R = \gamma \cap \{\lambda: |\lambda| \leq R\}$ with counterclockwise circuit, and also consider the contour $\gamma_R^0 = \gamma_R \cup \{\lambda: |\lambda| = R\}$ with clockwise circuit (see fig 2.7.2). By Cauchy‘s integral formula [12, p.84],

$$P_{1k}(x,\lambda) - \delta_{1k} = \frac{1}{2\pi i}\int_{\gamma_R^0} \frac{P_{1k}(x,\mu)-\delta_{1k}}{\lambda-\mu}\,d\mu, \quad \lambda \in int\,\gamma_R^0,$$

$$\frac{P_{jk}(x,\lambda)-P_{jk}(x,\mu)}{\lambda-\mu} = \frac{1}{2\pi i}\int_{\gamma_R^0} \frac{P_{jk}(x,\xi)}{(\lambda-\xi)(\xi-\mu)}\,d\xi, \quad \lambda,\mu \in int\,\gamma_R^0.$$

Using (2.7.109) we get

$$\lim_{R\to\infty}\int_{|\mu|=R} \frac{P_{1k}(x,\mu)-\delta_{1k}}{\lambda-\mu}\,d\mu = 0, \quad \lim_{R\to\infty}\int_{|\xi|=R} \frac{P_{jk}(x,\xi)}{(\lambda-\xi)(\xi-\mu)}\,d\xi = 0,$$

and consequently

$$P_{1k}(x,\lambda) = \delta_{1k} + \frac{1}{2\pi i}\int_\gamma \frac{P_{1k}(x,\mu)}{\lambda-\mu}\,d\mu, \quad \lambda \in J_\gamma, \tag{2.7.121}$$

$$\frac{P_{jk}(x,\lambda)-P_{jk}(x,\mu)}{\lambda-\mu} = \frac{1}{2\pi i}\int_\gamma \frac{P_{jk}(x,\xi)}{(\lambda-\xi)(\xi-\mu)}\,d\xi, \quad \lambda,\mu \in J_\gamma. \tag{2.7.122}$$

Here (and everywhere below, where necessary) the integral is understood in the principal value sense: $\int_\gamma = \lim_{R\to\infty}\int_{\gamma_R}$.

By virtue of (2.7.108) and (2.7.121),

$$\varphi(x,\lambda)=\tilde\varphi(x,\lambda)+\frac{1}{2\pi i}\int_\gamma \frac{\tilde\varphi(x,\lambda)P_{11}(x,\mu)+\tilde\varphi'(x,\lambda)P_{12}(x,\mu)}{\lambda-\mu}\,d\mu,\quad \lambda\in J_\gamma.$$

Taking (2.7.107) into account we get

$$\varphi(x,\lambda)=\tilde\varphi(x,\lambda)+\frac{1}{2\pi i}\int_\gamma(\tilde\varphi(x,\lambda)(\varphi(x,\mu)\tilde\Phi'(x,\mu)-\Phi(x,\mu)\tilde\varphi'(x,\mu))+$$

$$\tilde\varphi'(x,\lambda)(\Phi(x,\mu)\tilde\varphi(x,\mu)-\varphi(x,\mu)\tilde\Phi(x,\mu))\frac{d\mu}{\lambda-\mu}.$$

In view of (2.7.88), this yields (2.7.118), since the terms with $S(x,\mu)$ vanish by Cauchy's theorem. Using (2.7.122), (2.7.107)-(2.7.109) and (2.7.112), we arrive at

$$D(x,\lambda,\mu)-\tilde D(x,\lambda,\mu)=$$

$$\frac{1}{2\pi i}\int_\gamma\Big(\frac{\langle\tilde\varphi(x,\lambda),\tilde\Phi(x,\xi)\rangle\langle\varphi(x,\xi),\varphi(x,\mu)\rangle}{(\lambda-\xi)(\xi-\mu)}$$

$$-\frac{\langle\tilde\varphi(x,\lambda),\tilde\varphi(x,\xi)\rangle\langle\Phi(x,\xi),\varphi(x,\mu)\rangle}{(\lambda-\xi)(\xi-\mu)}\Big)\,d\xi.$$

In view of (2.7.88) and (2.7.112), this yields (2.7.119). □

Analogously one can obtain the relation

$$\tilde\Phi(x,\lambda)=\Phi(x,\lambda)+\frac{1}{2\pi i}\int_\gamma\frac{\langle\tilde\Phi(x,\lambda),\tilde\varphi(x,\mu)\rangle}{\lambda-\mu}\hat M(\mu)\varphi(x,\mu)\,d\mu,\ \lambda\in J_\gamma. \tag{2.7.123}$$

Let us consider the Banach space $C(\gamma)$ of continuous bounded functions $z(\lambda)$, $\lambda\in\gamma$, with the norm $\|z\|=\sup_{\lambda\in\gamma}|z(\lambda)|$.

Theorem 2.7.18. *For each fixed* $x\ge 0$, *the main equation* (2.7.118) *has a unique solution* $\varphi(x,\lambda)\in C(\gamma)$.

Proof. For a fixed $x\ge 0$, we consider the following linear bounded operators in $C(\gamma)$:

$$\tilde A z(\lambda)=z(\lambda)+\frac{1}{2\pi i}\int_\gamma \tilde r(x,\lambda,\mu)z(\mu)\,d\mu,$$

$$Az(\lambda)=z(\lambda)-\frac{1}{2\pi i}\int_\gamma r(x,\lambda,\mu)z(\mu)\,d\mu.$$

Then

$$\tilde A A z(\lambda)=z(\lambda)+\frac{1}{2\pi i}\int_\gamma\tilde r(x,\lambda,\mu)z(\mu)\,d\mu-\frac{1}{2\pi i}\int_\gamma r(x,\lambda,\mu)z(\mu)\,d\mu$$

$$-\frac{1}{2\pi i}\int_\gamma\tilde r(x,\lambda,\xi)\Big(\frac{1}{2\pi i}\int_\gamma r(x,\xi,\mu)z(\mu)\,d\mu\Big)\,d\xi$$

$$= z(\lambda) - \frac{1}{2\pi i}\int_\gamma \Big(r(x,\lambda,\mu) - \tilde r(x,\lambda,\mu) + \frac{1}{2\pi i}\int_\gamma \tilde r(x,\lambda,\xi) r(x,\xi,\mu)\,d\xi\Big) z(\mu)\,d\mu.$$

By virtue of (2.7.119) this yields

$$\tilde A A z(\lambda) = z(\lambda), \quad z(\lambda) \in C(\gamma).$$

Interchanging places for L and $\tilde L$, we obtain analogously $A\tilde A z(\lambda) = z(\lambda)$. Thus, $\tilde A A = A\tilde A = E$, where E is the identity operator. Hence the operator $\tilde A$ has a bounded inverse operator, and the main equation (2.7.118) is uniquely solvable for each fixed $x \ge 0$. □

Denote

$$\varepsilon_0(x) = \frac{1}{2\pi i}\int_\gamma \tilde\varphi(x,\mu)\varphi(x,\mu)\hat M(\mu)\,d\mu, \quad \varepsilon(x) = -2\varepsilon_0'(x). \tag{2.7.124}$$

Theorem 2.7.19. *The following relations hold*

$$q(x) = \tilde q(x) + \varepsilon(x), \tag{2.7.125}$$

$$h = \tilde h - \varepsilon_0(0). \tag{2.7.126}$$

Proof. Differentiating (2.7.118) twice with respect to x and using (2.7.112) and (2.7.124) we get

$$\tilde\varphi'(x,\lambda) - \varepsilon_0(x)\tilde\varphi(x,\lambda) = \varphi'(x,\lambda) + \frac{1}{2\pi i}\int_\gamma \tilde r(x,\lambda,\mu)\varphi'(x,\mu)\,d\mu, \tag{2.7.127}$$

$$\begin{aligned}\tilde\varphi''(x,\lambda) &= \varphi(x,\lambda) + \frac{1}{2\pi i}\int_\gamma \tilde r(x,\lambda,\mu)\varphi''(x,\mu)\,d\mu \\ &+ \frac{1}{2\pi i}\int_\gamma 2\tilde\varphi(x,\lambda)\tilde\varphi(x,\mu)\hat M(\mu)\varphi'(x,\mu)\,d\mu \\ &+ \frac{1}{2\pi i}\int_\gamma (\tilde\varphi(x,\lambda)\tilde\varphi(x,\mu))'\hat M(\mu)\varphi(x,\mu)\,d\mu.\end{aligned} \tag{2.7.128}$$

In (2.7.128) we replace the second derivatives using equation (2.7.30), and then we replace $\varphi(x,\lambda)$ using (2.7.118). This yields

$$\begin{aligned}\tilde q(x)\tilde\varphi(x,\lambda) &= q(x)\tilde\varphi(x,\lambda) + \frac{1}{2\pi i}\int_\gamma \langle \varphi(x,\lambda), \varphi(x,\mu)\rangle \hat M(\mu)\varphi(x,\mu)\,d\mu \\ &+ \frac{1}{2\pi i}\int_\gamma 2\tilde\varphi(x,\lambda)\tilde\varphi(x,\mu)\hat M(\mu)\varphi'(x,\mu)\,d\mu \\ &+ \frac{1}{2\pi i}\int_\gamma (\tilde\varphi(x,\lambda)\tilde\varphi(x,\mu))'\hat M(\mu)\varphi(x,\mu)\,d\mu.\end{aligned}$$

After canceling terms with $\tilde\varphi'(x,\lambda)$ we arrive at (2.7.125). Taking $x = 0$ in (2.7.127) we get (2.7.126). □

Thus, we obtain the following algorithm for the solution of the inverse problem.

Algorithm 2.7.2. *Let the function* $M(\lambda)$ *be given. Then*
(1) *Choose* $\tilde{L} \in V_N$ *such that* (2.7.110) *holds.*
(2) *Find* $\varphi(x,\lambda)$ *by solving equation* (2.7.118).
(3) *Construct* $q(x)$ *and* h *via* (2.7.124) – (2.7.126).

Let us now formulate necessary and sufficient conditions for the solvability of the inverse problem. Denote in the sequel by $\mathbf{W}$ the set of functions $M(\lambda)$ such that
(i) the functions $M(\lambda)$ are analytic in Π with the exception of an at most countable bounded set Λ' of poles, and are continuous in Π_1 with the exception of bounded set Λ (in general, Λ and Λ' are different for each function $M(\lambda)$);
(ii) for $|\lambda| \to \infty$, (2.7.91) holds.

Theorem 2.7.20. *For a function* $M(\lambda) \in \mathbf{W}$ *to be the Weyl function for a certain* $L \in V_N$, *it is necessary and sufficient that the following conditions hold:*
1) (*Asymptotics*) *There exists* $\tilde{L} \in V_N$ *such that* (2.7.110) *holds;*
2) (*Condition S*) *For each fixed* $x \geq 0$, *equation* (2.7.118) *has a unique solution* $\varphi(x,\lambda) \in C(\gamma)$;
3) $\varepsilon(x) \in W_N$, *where the function* $\varepsilon(x)$ *is defined by* (2.7.124).
Under these conditions $q(x)$ *and* h *are constructed via* (2.7.125) – (2.7.126).

As it is shown in Example 2.7.3, conditions 2) and 3) are essential and cannot be omitted. On the other hand, in [27, Sec. 2.3] we provide classes of operators for which the unique solvability of the main equation can be proved.

The *necessity* part of Theorem 2.7.20 was proved above. We prove now the *sufficiency*. Let a function $M(\lambda) \in \mathbf{W}$, satisfying the hypothesis of Theorem 2.7.20, be given, and let $\varphi(x,\lambda)$ be the solution of the main equation (2.7.118). Then (2.7.118) gives us the analytic continuation of $\varphi(x,\lambda)$ to the whole λ- plane, and for each fixed $x \geq 0$, the function $\varphi(x,\lambda)$ is entire in λ of order 1/2. Using Lemma 2.7.1 one can show that the functions $\varphi^{(\nu)}(x,\lambda)$, $\nu = 0,1$, are absolutely continuous with respect to x on compact sets, and

$$|\varphi^{(\nu)}(x,\lambda)| \leq C|\rho|^{\nu} \exp(|\tau|x). \tag{2.7.129}$$

We construct the function $\Phi(x,\lambda)$ via (2.7.123), and $L = L(q(x),h)$ via (2.7.125)-(2.7.126). Obviously, $L \in V_N$.

Lemma 2.7.9. *The following relations hold*

$$\ell\varphi(x,\lambda) = \lambda\varphi(x,\lambda), \quad \ell\Phi(x,\lambda) = \lambda\Phi(x,\lambda).$$

Proof. For simplicity, let

$$\int_{\lambda^*}^{\infty} \rho|\hat{V}(\lambda)|\,d\lambda < \infty$$

(the general case requires minor modifications). Then (2.7.129) is valid for $\nu = 0,1,2$. Differentiating (2.7.118) twice with respect to x we obtain (2.7.127) and (2.7.128). It follows from (2.7.128) and (2.7.118) that

$$-\tilde{\varphi}''(x,\lambda) + q(x)\tilde{\varphi}(x,\lambda) = \ell\varphi(x,\lambda) + \frac{1}{2\pi i}\int_{\gamma} \tilde{r}(x,\lambda,\mu)\ell\varphi(x,\mu)\,d\mu$$

$$+\frac{1}{2\pi i}\int_\gamma \langle\tilde\varphi(x,\lambda),\tilde\varphi(x,\mu)\rangle\hat M(\mu)\varphi(x,\mu)\,d\mu$$

$$-2\tilde\varphi(x,\lambda)\frac{1}{2\pi i}\int_\gamma (\tilde\varphi(x,\lambda)\tilde\varphi(x,\mu))'\hat M(\mu)\varphi(x,\mu)\,d\mu.$$

Taking (2.7.125) into account we get

$$\tilde\ell\tilde\varphi(x,\lambda)=\ell\varphi(x,\lambda)+\frac{1}{2\pi i}\int_\gamma \tilde r(x,\lambda,\mu)\ell\varphi(x,\mu)\,d\mu$$

$$+\frac{1}{2\pi i}\int_\gamma \langle\tilde\varphi(x,\lambda),\tilde\varphi(x,\mu)\rangle\hat M(\mu)\varphi(x,\mu)\,d\mu. \tag{2.7.130}$$

Using (2.7.123) we calculate similarly

$$\tilde\Phi'(x,\lambda)-\varepsilon_0(x)\tilde\Phi(x,\lambda)=\Phi'(x,\lambda)$$

$$+\frac{1}{2\pi i}\int_\gamma \frac{\langle\tilde\Phi(x,\lambda),\tilde\varphi(x,\mu)\rangle}{\lambda-\mu}\hat M(\mu)\varphi'(x,\mu)\,d\mu, \tag{2.7.131}$$

$$\tilde\ell\tilde\Phi(x,\lambda)=\ell\Phi(x,\lambda)+\frac{1}{2\pi i}\int_\gamma \frac{\langle\tilde\Phi(x,\lambda),\tilde\varphi(x,\mu)\rangle}{\lambda-\mu}\hat M(\mu)\ell\varphi(x,\mu)\,d\mu$$

$$+\frac{1}{2\pi i}\int_\gamma \langle\tilde\Phi(x,\lambda),\tilde\varphi(x,\mu)\rangle\hat M(\mu)\varphi(x,\mu)\,d\mu. \tag{2.7.132}$$

It follows from (2.7.130) that

$$\lambda\tilde\varphi(x,\lambda)=\ell\varphi(x,\lambda)+\frac{1}{2\pi i}\int_\gamma \tilde r(x,\lambda,\mu)\ell\varphi(x,\mu)\,d\mu$$

$$+\frac{1}{2\pi i}\int_\gamma (\lambda-\mu)\tilde r(x,\lambda,\mu)\varphi(x,\mu)\,d\mu.$$

Taking (2.7.118) into account we deduce for a fixed $x\ge 0$,

$$\eta(x,\lambda)+\frac{1}{2\pi i}\int_\gamma \tilde r(x,\lambda,\mu)\eta(x,\mu)\,d\mu=0,\quad \lambda\in\gamma, \tag{2.7.133}$$

where $\eta(x,\lambda):=\ell\varphi(x,\lambda)-\lambda\varphi(x,\lambda)$. According to (2.7.129) we have

$$|\eta(x,\lambda)|\le C_x|\rho|^2,\quad \lambda\in\gamma. \tag{2.7.134}$$

By virtue of (2.7.133), (2.7.112) and (2.7.113),

$$|\eta(x,\lambda)|\le C_x\Big(1+\int_{\lambda^*}^\infty \frac{|\hat V(\mu)|}{|\rho-\theta|+1}|\eta(x,\mu)|\,d\mu\Big), \tag{2.7.135}$$

$$\lambda\in\gamma,\ \theta>0,\ Re\,\rho\ge 0.$$

Substituting (2.7.134) into the right-hand side of (2.7.135) we get

$$|\eta(x,\lambda)|\le C_x\Big(1+\int_{\lambda^*}^\infty \frac{\theta^2|\hat V(\mu)|}{|\rho-\theta|+1}\,d\mu\Big),\quad \lambda\in\gamma,\ \theta>0,\ Re\,\rho\ge 0.$$

Since

$$\frac{\theta}{\rho(|\rho-\theta|+1)} \le 1 \ \text{ for } \ \theta,\rho \ge 1,$$

this yields

$$|\eta(x,\lambda)| \le C_x|\rho|, \quad \lambda \in \gamma. \tag{2.7.136}$$

Using (2.7.136) instead of (2.7.134) and repeating the preceding arguments we infer

$$|\eta(x,\lambda)| \le C_x, \quad \lambda \in \gamma.$$

According to Condition S of Theorem 2.7.20, the homogeneous equation (2.7.133) has only the trivial solution $\eta(x,\lambda) \equiv 0$. Consequently,

$$\ell\varphi(x,\lambda) = \lambda\varphi(x,\lambda). \tag{2.7.137}$$

It follows from (2.7.132) and (2.7.137) that

$$\lambda\tilde{\Phi}(x,\lambda) = \ell\Phi(x,\lambda) + \frac{1}{2\pi i}\int_\gamma \frac{\langle\tilde{\Phi}(x,\lambda),\tilde{\varphi}(x,\mu)\rangle}{\lambda-\mu}\hat{M}(\mu)\mu\varphi(x,\mu)\,d\mu$$

$$+\frac{1}{2\pi i}\int_\gamma(\lambda-\mu)\frac{\langle\tilde{\Phi}(x,\lambda),\tilde{\varphi}(x,\mu)\rangle}{\lambda-\mu}\hat{M}(\mu)\varphi(x,\mu)\,d\mu.$$

Together with (2.7.123) this yields $\ell\Phi(x,\lambda) = \lambda\Phi(x,\lambda)$. □

Lemma 2.7.10. *The following relations hold*

$$\varphi(0,\lambda) = 1,\ \varphi'(0,\lambda) = h. \tag{2.7.138}$$

$$U(\Phi) = 1, \quad \Phi(0,\lambda) = M(\lambda), \tag{2.7.139}$$

$$\Phi(x,\lambda) = O(\exp(i\rho x)),\ x \to \infty. \tag{2.7.140}$$

Proof. Taking $x = 0$ in (2.7.118) and (2.7.127) and using (2.7.126) we get

$$\varphi(0,\lambda) = \tilde{\varphi}(0,\lambda) = 1,$$

$$\varphi'(0,\lambda) = \tilde{\varphi}'(0,\lambda) - \varepsilon_0(0)\tilde{\varphi}(0,\lambda) = \tilde{h} + h - \tilde{h} = h,$$

i.e. (2.7.138) is valid. Using (2.7.123) and (2.7.131) we calculate

$$\Phi(0,\lambda) = \tilde{\Phi}(0,\lambda) + \frac{1}{2\pi i}\int_\gamma \frac{\hat{M}(\mu)}{\lambda-\mu}\,d\mu, \tag{2.7.141}$$

$$\Phi'(0,\lambda) = \tilde{\Phi}'(0,\lambda) - \tilde{\Phi}(0,\lambda)\varepsilon_0(0) + \frac{h}{2\pi i}\int_\gamma \frac{\hat{M}(\mu)}{\lambda-\mu}\,d\mu.$$

Consequently,

$$U(\Phi) = \Phi'(0,\lambda) - h\Phi(0,\lambda) = \tilde{\Phi}'(0,\lambda) - (\varepsilon_0(0)+h)\tilde{\Phi}(0,\lambda) =$$

$$\tilde{\Phi}'(0,\lambda) - \tilde{h}\tilde{\Phi}(0,\lambda) = \tilde{U}(\tilde{\Phi}) = 1.$$

Furthermore, since $\langle y,z\rangle = yz' - y'z,$ we rewrite (2.7.123) in the form

$$\Phi(x,\lambda) = \tilde\Phi(x,\lambda)$$
$$+\frac{1}{2\pi i}\int_\gamma \frac{\tilde\Phi'(x,\lambda)\tilde\varphi(x,\mu) - \tilde\Phi(x,\lambda)\tilde\varphi'(x,\mu)}{\lambda-\mu}\hat M(\mu)\varphi(x,\mu)\,d\mu, \tag{2.7.142}$$

where $\lambda \in J_\gamma$. The function $\varphi(x,\lambda)$ is the solution of the Cauchy problem (2.7.137)-(2.7.138). Therefore, according to (2.7.104),

$$|\varphi^{(\nu)}(x,\mu)| \le C|\theta|^\nu, \quad \mu = \theta^2 \in \gamma,\ x \ge 0,\ \nu = 0,1. \tag{2.7.143}$$

Moreover, the estimates (2.7.104)-(2.7.105) are valid for $\tilde\varphi(x,\lambda)$ and $\tilde\Phi(x,\lambda),$ i.e.

$$|\tilde\varphi^{(\nu)}(x,\mu)| \le C|\theta|^\nu, \quad \mu = \theta^2 \in \gamma,\ x \ge 0,\ \nu = 0,1, \tag{2.7.144}$$

$$|\tilde\Phi^{(\nu)}(x,\lambda)| \le C|\rho|^{\nu-1}\exp(-|Im\,\rho|x), \quad x \ge 0,\ \rho \in \Omega. \tag{2.7.145}$$

By virtue of (2.7.91),

$$\hat M(\lambda) = O\Big(\frac{1}{\lambda}\Big), \quad |\rho| \to \infty,\ \rho \in \Omega. \tag{2.7.146}$$

Fix $\lambda \in J_\gamma$. Taking (2.7.143)-(2.7.146) into account we get from (2.7.142) that

$$|\Phi(x,\lambda)\exp(-i\rho x)| \le C\Big(1 + \int_{\rho^*}^\infty \frac{d\theta}{\theta|\lambda-\mu|}\Big) \le C_1,$$

i.e. (2.7.140) is valid.

Furthermore, it follows from (2.7.141) that

$$\Phi(0,\lambda) = \tilde M(\lambda) + \frac{1}{2\pi i}\int_\gamma \frac{\hat M(\mu)}{\lambda-\mu}\,d\mu.$$

By Cauchy's integral formula

$$\hat M(\lambda) = \frac{1}{2\pi i}\int_{\gamma_R^0} \frac{\hat M(\mu)}{\lambda-\mu}\,d\mu, \quad \lambda \in int\,\gamma_R^0.$$

Since

$$\lim_{R\to\infty}\frac{1}{2\pi i}\int_{|\mu|=R} \frac{\hat M(\mu)}{\lambda-\mu}\,d\mu = 0,$$

we get

$$\hat M(\lambda) = \frac{1}{2\pi i}\int_\gamma \frac{\hat M(\mu)}{\lambda-\mu}\,d\mu, \quad \lambda \in J_\gamma.$$

Consequently, $\Phi(0,\lambda) = \tilde M(\lambda) + \hat M(\lambda) = M(\lambda),$ i.e. (2.7.139) is valid. □

Thus, $\Phi(x,\lambda)$ is the Weyl solution, and $M(\lambda)$ is the Weyl function for the constructed pair $L(q(x),h),$ and Theorem 2.7.20 is proved. □

Example 2.7.3. Let $\tilde q(x) = 0$ and $\tilde h = 0.$ Then $\tilde M(\lambda) = \dfrac{1}{i\rho}.$ Consider the function

$$M(\lambda) = \tilde M(\lambda) + \frac{a}{\lambda-\lambda_0},$$

where a and λ_0 are complex numbers. Then the main equation (2.7.118) becomes

$$\tilde{\varphi}(x,\lambda_0) = F(x)\varphi(x,\lambda_0),$$

where

$$\tilde{\varphi}(x,\lambda_0) = \cos\rho_0 x, \quad F(x) = 1 + a\int_0^x \cos^2\rho_0 t\,dt, \quad \lambda_0 = \rho_0^2.$$

The solvability condition for the main equation takes the form

$$F(x) \neq 0 \ \text{ for all } \ x \geq 0,$$

and the function $\varepsilon(x)$ can be found by the formula

$$\varepsilon(x) = \frac{2a\rho_0 \sin 2\rho_0 x}{F(x)} + \frac{2a^2\cos^4\rho_0 x}{F^2(x)}.$$

Case 1. Let $\lambda_0 = 0.$ Then

$$F(x) = 1 + ax,$$

and the solvability condition is equivalent to the condition

$$a \notin (-\infty, 0).$$

If this is fulfilled then $M(\lambda)$ is the Weyl function for L of the form (2.7.30)-(2.7.31) with

$$q(x) = \frac{2a^2}{(1+ax)^2}, \quad h = -a,$$

$$\varphi(x,\lambda) = \cos\rho x - \frac{a}{1+ax}\cdot\frac{\sin\rho x}{\rho},\ e(x,\rho) = \exp(i\rho x)\Big(1 - \frac{a}{i\rho(1+ax)}\Big),$$

$$\Delta(\rho) = i\rho, \quad V(\lambda) = \frac{1}{\pi\rho}, \quad \varphi(x,0) = \frac{1}{1+ax}.$$

If $a < 0,$ then the solvability condition is not fulfilled, and the function $M(\lambda)$ is not a Weyl function.

Case 2. Let $\lambda_0 \neq 0$ be a real number, and let $a > 0.$ Then $F(x) \geq 1,$ and the solvability condition is fulfilled. But in this case $\varepsilon(x) \notin L(0,\infty),$ i.e. $\varepsilon(x) \notin W_N$ for any $N \geq 0.$

The Gelfand-Levitan method. For Sturm-Liouville operators on a finite interval, the Gelfand-Levitan method was presented in subsection 2. For the case of the half-line there are similar results. Therefore here we confine ourselves to the derivation of the Gelfand-Levitan equation. For further discussion see [15], [16] and [25].

Consider the differential equation and the linear form $L = L(q(x),h)$ of the form (2.7.30)-(2.7.31). Let $\tilde{q}(x) = 0,\ \tilde{h} = 0.$ Denote

$$F(x,t) = \frac{1}{2\pi i}\int_\gamma \cos\rho x\cos\rho t\hat{M}(\lambda)\,d\lambda, \tag{2.7.147}$$

where γ is the contour defined above (see fig. 2.7.1). We note that by virtue of (2.7.111) and (2.7.111),

$$\frac{1}{2\pi i}\int_{\gamma''}\cos\rho x\cos\rho t\hat{M}(\lambda)\,d\lambda=\int_{\lambda^*}^{\infty}\cos\rho x\cos\rho t\hat{V}(\lambda)\,d\lambda<\infty.$$

Let $G(x,t)$ be taken from (2.7.4), and let $H(x,t)$ be the kernel of the operator $E+H:=(E+G)^{-1}$, i.e.

$$\cos\rho x=\varphi(x,\lambda)+\int_0^x H(x,t)\varphi(t,\lambda)\,dt. \tag{2.7.148}$$

Theorem 2.7.21. For each fixed x, the function $G(x,t)$ satisfies the following linear integral equation

$$G(x,t)+F(x,t)+\int_0^x G(x,s)F(s,t)ds=0,\quad 0<t<x. \tag{2.7.149}$$

Equation (2.7.149) *is the Gelfand-Levitan equation.*

Proof. Using (2.7.4) and (1.7.148) we calculate

$$\frac{1}{2\pi i}\int_{\gamma_R}\varphi(x,\lambda)\cos\rho tM(\lambda)\,d\lambda=\frac{1}{2\pi i}\int_{\gamma_R}\cos\rho x\cos\rho tM(\lambda)\,d\lambda$$

$$+\frac{1}{2\pi i}\int_{\gamma_R}\left(\int_0^x G(x,s)\cos\rho s\,ds\right)\cos\rho tM(\lambda)\,d\lambda,$$

$$\frac{1}{2\pi i}\int_{\gamma_R}\varphi(x,\lambda)\cos\rho tM(\lambda)\,d\lambda=\frac{1}{2\pi i}\int_{\gamma_R}\varphi(x,\lambda)\varphi(t,\lambda)M(\lambda)\,d\lambda$$

$$+\frac{1}{2\pi i}\int_{\gamma_R}\left(\int_0^t H(t,s)\varphi(s,\lambda)\,ds\right)\varphi(x,\lambda)M(\lambda)\,d\lambda,$$

where $\gamma_R=\gamma\cap\{\lambda:\ |\lambda|\le R\}$. This yields

$$\Phi_R(x,t)=I_{R1}(x,t)+I_{R2}(x,t)+I_{R3}(x,t)+I_{R4}(x,t),$$

where

$$\Phi_R(x,t)=\frac{1}{2\pi i}\int_{\gamma_R}\varphi(x,\lambda)\varphi(t,\lambda)M(\lambda)\,d\lambda-\frac{1}{2\pi i}\int_{\gamma_R}\cos\rho x\cos\rho t\tilde{M}(\lambda)\,d\lambda,$$

$$I_{R1}(x,t)=\frac{1}{2\pi i}\int_{\gamma_R}\cos\rho x\cos\rho t\hat{M}(\lambda)d\lambda,$$

$$I_{R2}(x,t)=\int_0^x G(x,s)\left(\frac{1}{2\pi i}\int_{\gamma_R}\cos\rho t\cos\rho s\hat{M}(\lambda)d\lambda\right)ds,$$

$$I_{R3}(x,t)=\frac{1}{2\pi i}\int_{\gamma_R}\cos\rho t\left(\int_0^x G(x,s)\cos\rho s\,ds\right)\tilde{M}(\lambda)d\lambda,$$

$$I_{R4}(x,t)=-\frac{1}{2\pi i}\int_{\gamma_R}\varphi(x,\lambda)\left(\int_0^t H(t,s)\varphi(s,\lambda)\,ds\right)M(\lambda)\,d\lambda.$$

Let $\xi(t),\ t \geq 0$ be a twice continuously differentiable function with compact support. By Theorem 2.7.15,

$$\lim_{R\to\infty}\int_0^\infty \xi(t)\Phi_R(x,t)dt = 0,\ \lim_{R\to\infty}\int_0^\infty \xi(t)I_{R1}(x,t)dt = \int_0^\infty \xi(t)F(x,t)dt,$$

$$\lim_{R\to\infty}\int_0^\infty \xi(t)I_{R2}(x,t)dt = \int_0^\infty \xi(t)\left(\int_0^x G(x,s)F(s,t)ds\right)dt,$$

$$\lim_{R\to\infty}\int_0^\infty \xi(t)I_{R3}(x,t)dt = \int_0^x \xi(t)G(x,t)dt,$$

$$\lim_{R\to\infty}\int_0^\infty \xi(t)I_{R4}(x,t)dt = -\int_x^\infty \xi(t)H(t,x)dt.$$

Put $G(x,t) = H(x,t) = 0$ for $x < t$. In view of the arbitrariness of $\xi(t)$, we derive

$$G(x,t) + F(x,t) + \int_0^x G(x,s)F(s,t)ds - H(t,x) = 0.$$

For $t < x$, this gives (2.7.149). □

Thus, in order to solve the inverse problem of recovering L from the Weyl function $M(\lambda)$ one can calculate $F(x,t)$ by (2.7.147), find $G(x,t)$ by solving the Gelfand-Levitan equation (2.7.149) and construct $q(x)$ and h by (2.7.27).

Remark 2.7.9. We show the connection between the Gelfand-Levitan equation and the main equation of inverse problem (2.7.118). For this purpose we use the cosine Fourier transform. Let $\tilde{q}(x) = \tilde{h} = 0$. Then $\tilde{\varphi}(x,\lambda) = \cos\sqrt{\lambda}x$. Multiplying (2.7.149) by $\cos\sqrt{\lambda}t$ and integrating with respect to t, we obtain

$$\int_0^x G(x,t)\cos\sqrt{\lambda}t\,dt + \int_0^x \cos\sqrt{\lambda}t\left(\frac{1}{2\pi i}\int_\gamma \cos\sqrt{\mu}x\cos\sqrt{\mu}t\hat{M}(\mu)\,d\mu\right)dt$$

$$+\int_0^x \cos\sqrt{\lambda}t\int_0^x G(x,s)\left(\frac{1}{2\pi i}\int_\gamma \cos\sqrt{\mu}t\cos\sqrt{\mu}s\hat{M}(\mu)\,d\mu\right)ds = 0.$$

Using (2.7.4) we arrive at (2.7.118).

Remark 2.7.10. If $q(x)$ and h are real, and $(1+x)q(x) \in L(0,\infty)$, then (2.7.147) takes the form

$$F(x,t) = \int_{-\infty}^\infty \cos\rho x\cos\rho t\,d\hat{\sigma}(\lambda),$$

where $\hat{\sigma} = \sigma - \tilde{\sigma}$, and σ and $\tilde{\sigma}$ are the spectral functions of L and $\tilde{L}$ respectively.

5. The generalized Weyl function

Let us consider the differential equation and the linear form $L = L(q(x),h)$:

$$\ell y := -y'' + q(x)y = \lambda y, \quad x > 0,$$

$$U(y) := y'(0) - hy(0).$$

In this subsection we study the inverse spectral problem for L in the case when $q(x)$ is a locally integrable complex-valued function, and h is a complex number. We introduce the so-called generalized Weyl function as a main spectral characteristic.

For this purpose we define a space of generalized functions (distributions). Let D be the set of all integrable and bounded on the real line entire functions of exponential type with ordinary operations of addition and multiplication by complex numbers and with the following convergence: $z_k(\rho)$ is said to converge to $z(\rho)$ if the types σ_k of the functions $z_k(\rho)$ are bounded ($\sup\sigma_k < \infty$), and $\|z_k(\rho) - z(\rho)\|_{L(-\infty,\infty)} \to 0$ as $k \to \infty$. The linear manifold D with this convergence is our space of test functions.

Definition 2.7.2. All linear and continuous functionals

$$R: \ D \to \mathbf{C}, \ z(\rho) \mapsto R(z(\rho)) = (z(\rho), R),$$

are called generalized functions (GF). The set of these GF is denoted by D'. A sequence of GF $R_k \in D'$ converges to $R \in D'$, if $\lim (z(\rho), R_k) = (z(\rho), R)$, $k \to \infty$ for any $z(\rho) \in D$. A GF $R \in D'$ is called regular if it is determined by $R(\rho) \in L_\infty$ via

$$(z(\rho), R) = \int_{-\infty}^{\infty} z(\rho) R(\rho)\, d\rho.$$

Definition 2.7.3. Let a function $f(t)$ be locally integrable for $t > 0$ (i.e. it is integrable on every finite segment $[0, T]$). The GF $L_f(\rho) \in D'$ defined by the equality

$$(z(\rho), L_f(\rho)) := \int_0^{\infty} f(t) \Big(\int_{-\infty}^{\infty} z(\rho) \exp(i\rho t)\, d\rho \Big)\, dt, \ z(\rho) \in D, \tag{2.7.150}$$

is called the generalized Fourier-Laplace transform for the function $f(t)$.

Since $z(\rho) \in D$, we have

$$\int_{-\infty}^{\infty} |z(\rho)|^2\, d\rho \le \sup_{-\infty<\rho<\infty} |z(\rho)| \cdot \int_{-\infty}^{\infty} |z(\rho)|\, d\rho,$$

i.e. $z(\rho) \in L_2(-\infty, \infty)$. Therefore, by virtue of the Paley-Wiener theorem [4], the function

$$B(t) := \frac{1}{2\pi} \int_{-\infty}^{\infty} z(\rho) \exp(i\rho t)\, d\rho$$

is continuous and has compact support, i.e. there exists a $d > 0$ such that $B(t) = 0$ for $|t| > d$, and

$$z(\rho) = \int_{-d}^{d} B(t) \exp(-i\rho t)\, dt. \tag{2.7.151}$$

Consequently, the integral in (2.7.150) exists. We note that $f(t) \in L(0, \infty)$ implies

$$(z(\rho), L_f(\rho)) := \int_{-\infty}^{\infty} z(\rho) \Big(\int_0^{\infty} f(t) \exp(i\rho t)\, dt \Big)\, d\rho,$$

i.e. $L_f(\rho)$ is a regular GF (defined by $\displaystyle\int_0^{\infty} f(t) \exp(i\rho t)\, dt$) and coincides with the ordinary Fourier-Laplace transform for the function $f(t)$. Since

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1 - \cos\rho x}{\rho^2} \exp(i\rho t) d\rho = \begin{cases} x - t, & t < x, \\ 0, & t > x, \end{cases}$$

the following inversion formula is valid:

$$\int_0^x (x-t)f(t)dt = \left(\frac{1}{\pi}\cdot\frac{1-\cos\rho x}{\rho^2},\ L_f(\rho)\right). \tag{2.7.152}$$

Let now $u(x,t)$ be the solution of (2.6.1)-(2.6.2) with a locally integrable complex-valued function $q(x)$. Define $u(x,t)=0$ for $0<t<x$, and denote (with $\lambda=\rho^2$) $\Phi(x,\lambda):=-L_u(\rho)$, i.e.

$$(z(\rho),\Phi(x,\lambda)) = -\int_x^\infty u(x,t)\Big(\int_{-\infty}^\infty z(\rho)\exp(i\rho t)d\rho\Big)dt. \tag{2.7.153}$$

For $z(\rho)\in D$, $\rho^2 z(\rho)\in L(-\infty,\infty)$, $\nu=1,2$, we put

$$(z(\rho),(i\rho)^\nu\Phi(x,\lambda)) := ((i\rho)^\nu z(\rho),\Phi(x,\lambda)),$$

$$(z(\rho),\Phi^{(\nu)}(x,\lambda)) := \frac{d^\nu}{dx^\nu}(z(\rho),\Phi(x,\lambda)).$$

Theorem 2.7.22. *The following relations hold*

$$\ell\Phi(x,\lambda)=\lambda\Phi(x,\lambda),\quad U(\Phi)=1.$$

Proof. We calculate

$$(z(\rho),\ell\Phi(x,\lambda)) = (z(\rho),-\Phi''(x,\lambda)+q(x)\Phi(x,\lambda))$$

$$= -\int_{-\infty}^\infty (i\rho)z(\rho)\exp(i\rho x)d\rho - u_x(x,x)\int_{-\infty}^\infty z(\rho)\exp(i\rho x)d\rho$$

$$+\int_x^\infty \Big(u_{xx}(x,t)-q(x)u(x,t)\Big)\Big(\int_{-\infty}^\infty z(\rho)\exp(i\rho t)\,d\rho\Big)dt,$$

$$(z(\rho),\lambda\Phi(x,\lambda)) = -\int_{-\infty}^\infty (i\rho)z(\rho)\exp(i\rho x)d\rho - u_t(x,x)\int_{-\infty}^\infty z(\rho)\exp(i\rho x)d\rho$$

$$+\int_x^\infty u_{tt}(x,t)\Big(\int_{-\infty}^\infty z(\rho)\exp(i\rho t)\,d\rho\Big)dt.$$

Using $u_t(x,x)+u_x(x,x)=\dfrac{d}{dx}u(x,x)\equiv 0$, we infer $(z(\rho),\ell\Phi(x,\lambda)-\lambda\Phi(x,\lambda))=0$. Furthermore, since

$$(z(\rho),\Phi'(x,\lambda)) = \int_{-\infty}^\infty z(\rho)\exp(i\rho x)\,d\rho$$

$$-\int_x^\infty u_x(x,t)\Big(\int_{-\infty}^\infty z(\rho)\exp(i\rho t)\,d\rho\Big)dt,$$

we get

$$(z(\rho),U(\Phi)) = (z(\rho),\Phi'(0,\lambda)-h\Phi(0,\lambda)) = \int_{-\infty}^\infty z(\rho)\,d\rho.$$

□

Definition 2.7.4. The GF $\Phi(x,\lambda)$ is called the *generalized Weyl solution*, and the GF $M(\lambda):=\Phi(0,\lambda)$ is called the *generalized Weyl function (GWF)* for $L(q(x),h)$.

Note that if $q(x) \in L(0,\infty)$, then $|u(x,t)| \le C_1 \exp(C_2 t)$, and $\Phi(x,\lambda)$ and $M(\lambda)$ coincide with the ordinary Weyl solution and Weyl function (see [17, Ch.2]).

The inverse problem considered here is formulated as follows:

Inverse Problem 2.7.3. Given the GWF $M(\lambda)$, construct the potential $q(x)$ and the coefficient h.

Denote $r(t) := u(0,t)$. It follows from (2.7.153) that

$$(z(\rho), M(\lambda)) = -\int_0^\infty r(t)\Big(\int_{-\infty}^\infty z(\rho)\exp(i\rho t)\,d\rho\Big)\,dt,$$

i.e. $M(\lambda) = -L_r(\rho)$. In view of (2.7.152), we get by differentiation

$$r(t) = -\frac{d^2}{dt^2}\left(\frac{1}{\pi}\cdot\frac{1-\cos\rho t}{\rho^2},\, M(\lambda)\right), \tag{2.7.154}$$

and Inverse Problem 2.7.3 has been reduced to Inverse Problem 2.6.1 from the trace r considered in Section 2.6. Thus, the following theorems hold.

Theorem 2.7.23. *Let $M(\lambda)$ and $\tilde M(\lambda)$ be the GWF's for $L = L(q(x),h)$ and $\tilde L = L(\tilde q(x),\tilde h)$ respectively. If $M(\lambda) = \tilde M(\lambda)$, then $L = \tilde L$. Thus, the specification of the GWF uniquely determines the potential q and the coefficient h.*

Theorem 2.7.24. *Let $\varphi(x,\lambda)$ be the solution of the differential equation $\ell\varphi = \lambda\varphi$ under the initial conditions $\varphi(0,\lambda) = 1$, $\varphi'(0,\lambda) = h$. Then the following representation holds*

$$\varphi(x,\lambda) = \cos\rho x + \int_0^x G(x,t)\cos\rho t\,dt,$$

and the function $G(x,t)$ satisfies the integral equation

$$G(x,t) + F(x,t) + \int_0^x G(x,\tau)F(t,\tau)\,d\tau = 0, \quad 0 < t < x, \tag{2.7.155}$$

where

$$F(x,t) = \frac{1}{2}\Big(r'(t+x) + r'(t-x)\Big).$$

The function r is defined via (2.7.154), and $r \in D_2$. If $q \in D_N$ then $r \in D_{N+2}$. Moreover, for each fixed $x > 0$, the integral equation (2.7.155) is uniquely solvable.

Theorem 2.7.25. *For a generalized function $M \in D'$ to be the GWF for a certain $L(q(x),h)$ with $q \in D_N$, it is necessary and sufficient that*
1) *$r \in D_{N+2}$, $r(0) = 1$, where r is defined via (2.7.154);*
2) *for each $x > 0$, the integral equation (2.7.155) is uniquely solvable.*

The potential q and the coefficient h can be constructed by the following algorithm.

Algorithm 2.7.3. Let the GWF $M(\lambda)$ be given. Then
(1) Construct the function $r(t)$ by (2.7.154).

(2) Find the function $G(x,t)$ by solving the integral equation (2.7.155).
(3) Calculate $q(x)$ and h by

$$q(x) = 2\frac{dG(x,x)}{dx}, \quad h = G(0,0).$$

Let us now prove an expansion theorem for the case of locally integrable complex-valued potentials q.

Theorem 2.7.26. *Let $f(x) \in W_2$. Then, uniformly on compact sets,*

$$f(x) = \frac{1}{\pi}\Big(\varphi(x,\lambda)F(\lambda)(i\rho), M(\lambda)\Big), \tag{2.7.156}$$

where

$$F(\lambda) = \int_0^\infty f(t)\varphi(t,\lambda)\,dt. \tag{2.7.157}$$

Proof. First we assume that $q(x) \in L(0,\infty)$. Let $f(x) \in Q$, where $Q = \{f \in W_2 : U(f) = 0,\ \ell f \in L_2(0,\infty)\}$ (the general case when $f \in W_2$ requires small modifications).

Let $D^+ = \{z(\rho) \in D :\ \rho z(\rho) \in L_2(-\infty,\infty)\}$. Clearly, $z(\rho) \in D^+$ if and only if $B(t) \in W_2^1[-d,d]$ in (2.7.151). For $z(\rho) \in D^1$, integration by parts in (2.7.151) yields

$$z(\rho) = \int_{-d}^{d} B(t)\exp(-i\rho t)\,dt = \frac{1}{i\rho}\int_{-d}^{d} B'(t)\exp(-i\rho t)\,dt.$$

Using (2.7.157) we calculate

$$F(\lambda) = \frac{1}{\lambda}\int_0^\infty f(t)\Big(-\varphi''(t,\lambda) + q(t)\varphi(t,\lambda)\Big)\,dt = \frac{1}{\lambda}\int_0^\infty \varphi(t,\lambda)\ell f(t)\,dt,$$

and consequently $F(\lambda)(i\rho) \in D^+$. According to Theorem 2.7.15 we have

$$f(x) = \frac{1}{2\pi i}\int_\gamma \varphi(x,\lambda)F(\lambda)M(\lambda)\,d\lambda = -\frac{1}{\pi}\int_{\gamma_1} \varphi(x,\lambda)F(\lambda)(i\rho)M(\lambda)\,d\rho, \tag{2.7.158}$$

where the contour γ_1 in the ρ - plane is the image of γ under the mapping $\rho \to \lambda = \rho^2$. In view of Remark 2.6.4,

$$M(\lambda) = -\int_0^\infty r(t)\exp(i\rho t)\,dt, \quad |r(t)| \le C_1\exp(C_2 t). \tag{2.7.159}$$

Take $b > C_2$. Then, by virtue of Cauchy‘s theorem, (2.7.158)-(2.7.159) imply

$$f(x) = \frac{1}{\pi}\int_{-\infty+ib}^{\infty+ib} \varphi(x,\lambda)F(\lambda)(i\rho)\Big(-\int_0^\infty r(t)\exp(i\rho t)\,dt\Big)\,d\rho$$

$$= -\frac{1}{\pi}\int_0^\infty r(t)\Big(\int_{-\infty+ib}^{\infty+ib} \varphi(x,\lambda)F(\lambda)(i\rho)\exp(i\rho t)\,d\rho\Big)\,dt.$$

Using Cauchy‘s theorem again we get

$$f(x) = -\frac{1}{\pi}\int_0^\infty r(t)\Big(\int_{-\infty}^{\infty} \varphi(x,\lambda)F(\lambda)(i\rho)\exp(i\rho t)\,d\rho\Big)\,dt$$

$$= \frac{1}{\pi}\Big(\varphi(x,\lambda)F(\lambda)(i\rho), M(\lambda)\Big),$$

i.e. (2.7.156) is valid.

Let now $q(x)$ be a locally integrable complex-valued function. Denote

$$q_R(x) = \begin{cases} q(x), & 0 \le x \le R, \\ 0, & x > R. \end{cases}$$

Let $r_R(t)$ be the trace for the potential q_R. According to Remark 2.6.2,

$$r_R(t) = r(t) \ \text{ for } \ t \le 2R. \tag{2.7.160}$$

Since $q_R(x) \in L(0,\infty)$ we have by virtue of (2.7.156),

$$f(x) = -\frac{1}{\pi}\int_0^\infty r_R(t)\Big(\int_{-\infty}^\infty \varphi(x,\lambda)F(\lambda)(i\rho)\exp(i\rho t)\,d\rho\Big)\,dt.$$

Let $x \in [0,T]$ for a certain $T > 0$. Then there exists a $d > 0$ such that

$$f(x) = -\frac{1}{\pi}\int_0^d r_R(t)\Big(\int_{-\infty}^\infty \varphi(x,\lambda)F(\lambda)(i\rho)\exp(i\rho t)\,d\rho\Big)\,dt,\ 0 \le x \le T.$$

For sufficiently large R $(R > d/2)$ we have in view of (2.7.160),

$$f(x) = -\frac{1}{\pi}\int_0^d r(t)\Big(\int_{-\infty}^\infty \varphi(x,\lambda)F(\lambda)(i\rho)\exp(i\rho t)\,d\rho\Big)\,dt$$

$$= -\frac{1}{\pi}\int_0^\infty r(t)\Big(\int_{-\infty}^\infty \varphi(x,\lambda)F(\lambda)(i\rho)\exp(i\rho t)\,d\rho\Big)\,dt$$

$$= \frac{1}{\pi}\Big(\varphi(x,\lambda)F(\lambda)(i\rho), M(\lambda)\Big),$$

i.e. (2.7.156) is valid, and Theorem 2.7.26 is proved. □

2.8. Inverse Scattering on the Line

In this section the inverse scattering problem for the Sturm-Liouville operator on the line is considered. In subsection 1 we introduce the scattering data and study their properties. In subsection 2, using the transformation operator method, we give a derivation of the corresponding main equation and prove its unique solvability. We also provide an algorithm for the solution of the inverse scattering problem along with necessary and sufficient conditions for its solvability. In subsection 3 a class of reflectionless potentials, which is important for applications, is studied, and an explicit formula for constructing such potentials is given. We note that the inverse scattering problem for the Sturm-Liouville operator on the line was considered in the monographs [15]-[17].

1. Scattering data

Let us consider the differential equation

$$\ell y := -y'' + q(x)y = \lambda y, \quad -\infty < x < \infty. \tag{2.8.1}$$

Everywhere below in this section we will assume that the function $q(x)$ is real, and that

$$\int_{-\infty}^{\infty} (1+|x|)|q(x)|\,dx < \infty. \tag{2.8.2}$$

Let $\lambda = \rho^2,\ \rho = \sigma + i\tau,$ and let for definiteness $\tau := Im\rho \ge 0.$ Denote $\Omega_+ = \{\rho :\ Im\rho > 0\},$

$$Q_0^+(x) = \int_x^{\infty} |q(t)|\,dt, \quad Q_1^+(x) = \int_x^{\infty} Q_0^+(t)\,dt = \int_x^{\infty} (t-x)|q(t)|\,dt,$$

$$Q_0^-(x) = \int_{-\infty}^{x} |q(t)|\,dt, \quad Q_1^-(x) = \int_{-\infty}^{x} Q_0^-(t)\,dt = \int_{-\infty}^{x} (t-x)|q(t)|\,dt.$$

Clearly,

$$\lim_{x\to\pm\infty} Q_j^{\pm}(x) = 0.$$

The following theorem introduces the Jost solutions $e(x,\rho)$ and $g(x,\rho)$ with prescribed behavior in $\pm\infty$.

Theorem 2.8.1. *Equation* $(2.8.1)$ *has unique solutions* $y = e(x,\rho)$ *and* $y = g(x,\rho)$, *satisfying the integral equations*

$$e(x,\rho) = \exp(i\rho x) + \int_x^{\infty} \frac{\sin\rho(t-x)}{\rho} q(t)e(t,\rho)\,dt,$$

$$g(x,\rho) = \exp(-i\rho x) + \int_{-\infty}^{x} \frac{\sin\rho(x-t)}{\rho} q(t)g(t,\rho)\,dt.$$

The functions $e(x,\rho)$ *and* $g(x,\rho)$ *have the following properties:*

1) *For each fixed* $x,$ *the functions* $e^{(\nu)}(x,\rho)$ *and* $g^{(\nu)}(x,\rho)$ $(\nu = 0,1)$ *are analytic in* Ω_+ *and continuous in* $\overline{\Omega}_+$.

2) *For* $\nu = 0,1,$

$$\left.\begin{aligned} e^{(\nu)}(x,\rho) &= (i\rho)^{\nu}\exp(i\rho x)(1+o(1)), \quad x\to+\infty,\\ g^{(\nu)}(x,\rho) &= (-i\rho)^{\nu}\exp(-i\rho x)(1+o(1)), \quad x\to-\infty, \end{aligned}\right\} \tag{2.8.3}$$

uniformly in $\overline{\Omega}_+$. *Moreover, for* $\rho \in \overline{\Omega}_+$,

$$\left.\begin{aligned} |e(x,\rho)\exp(-i\rho x)| &\le \exp(Q_1^+(x)),\\ |e(x,\rho)\exp(-i\rho x) - 1| &\le Q_1^+(x)\exp(Q_1^+(x)),\\ |e'(x,\rho)\exp(-i\rho x) - i\rho| &\le Q_0^+(x)\exp(Q_1^+(x)), \end{aligned}\right\} \tag{2.8.4}$$

$$\left.\begin{aligned} |g(x,\rho)\exp(i\rho x)| &\le \exp(Q_1^-(x)),\\ |g(x,\rho)\exp(i\rho x)-1| &\le Q_1^-(x)\exp(Q_1^-(x)),\\ |g'(x,\rho)\exp(i\rho x)+i\rho| &\le Q_0^-(x)\exp(Q_1^-(x)). \end{aligned}\right\} \tag{2.8.5}$$

3) *For each fixed* $\rho\in\Omega_+$ *and each real* α, $e(x,\rho)\in L_2(\alpha,\infty)$, $g(x,\rho)\in L_2(-\infty,\alpha)$. *Moreover,* $e(x,\rho)$ *and* $g(x,\rho)$ *are the unique solutions of* (2.8.1) (*up to a multiplicative constant*) *having this property.*

4) *For* $|\rho|\to\infty$, $\rho\in\overline{\Omega}_+$, $\nu=0,1$,

$$\left.\begin{aligned} e^{(\nu)}(x,\rho) &= (i\rho)^\nu\exp(i\rho x)\Big(1+\frac{\omega^+(x)}{i\rho}+o\Big(\frac{1}{\rho}\Big)\Big),\\ g^{(\nu)}(x,\rho) &= (-i\rho)^\nu\exp(-i\rho x)\Big(1+\frac{\omega^-(x)}{i\rho}+o\Big(\frac{1}{\rho}\Big)\Big), \end{aligned}\right\} \tag{2.8.6}$$

$$\omega^+(x):=-\frac{1}{2}\int_x^\infty q(t)\,dt,\quad \omega^-(x):=-\frac{1}{2}\int_{-\infty}^x q(t)\,dt,$$

uniformly for $x\ge\alpha$ *and* $x\le\alpha$ *respectivrly.*

5) *For real* $\rho\ne 0$, *the functions* $\{e(x,\rho),e(x,-\rho)\}$ *and* $\{g(x,\rho),g(x,-\rho)\}$ *form fundamental systems of solutions for (2.8.1), and*

$$\langle e(x,\rho),e(x,-\rho)\rangle = -\langle g(x,\rho),g(x,-\rho)\rangle \equiv -2i\rho, \tag{2.8.7}$$

where $\langle y,z\rangle := yz'-y'z$.

6) *The functions* $e(x,\rho)$ *and* $g(x,\rho)$ *have the representations*

$$\left.\begin{aligned} e(x,\rho) &= \exp(i\rho x)+\int_x^\infty A^+(x,t)\exp(i\rho t)\,dt,\\ g(x,\rho) &= \exp(-i\rho x)+\int_{-\infty}^x A^-(x,t)\exp(-i\rho t)\,dt, \end{aligned}\right\} \tag{2.8.8}$$

where $A^\pm(x,t)$ *are real continuous functions, and*

$$A^+(x,x)=\frac{1}{2}\int_x^\infty q(t)dt,\quad A^-(x,x)=\frac{1}{2}\int_{-\infty}^x q(t)dt, \tag{2.8.9}$$

$$|A^\pm(x,t)|\le\frac{1}{2}Q_0^\pm\Big(\frac{x+t}{2}\Big)\exp\Big(Q_1^\pm(x)-Q_1^\pm\Big(\frac{x+t}{2}\Big)\Big). \tag{2.8.10}$$

The functions $A^\pm(x,t)$ *have first derivatives* $A_1^\pm:=\frac{\partial A^\pm}{\partial x}$, $A_2^\pm:=\frac{\partial A^\pm}{\partial t}$; *the functions*

$$A_i^\pm(x,t)\pm\frac{1}{4}q\Big(\frac{x+t}{2}\Big)$$

are absolutely continuous with respect to x *and* t, *and*

$$\Big|A_i^\pm(x,t)\pm\frac{1}{4}q\Big(\frac{x+t}{2}\Big)\Big|\le\frac{1}{2}Q_0^\pm(x)Q_0^\pm\Big(\frac{x+t}{2}\Big)\exp(Q_1^\pm(x)),\ i=1,2. \tag{2.8.11}$$

For the function $e(x,\rho)$, Theorem 2.8.1 was proved in Section 2.7 (see Theorems 2.7.7-2.7.9). For $g(x,\rho)$ the arguments are the same. Moreover, all assertions of Theorem 2.8.1 for $g(x,\rho)$ can be obtained from the corresponding assertions for $e(x,\rho)$ by the replacement $x \to -x$.

In the next lemma we describe properties of the Jost solutions $e_j(x,\rho)$ and $g_j(x,\rho)$ related to the potentials q_j, which approximate q.

Lemma 2.8.1. *If* $(1+|x|)|q(x)| \in L(a,\infty)$, $a > -\infty$, *and*

$$\lim_{j\to\infty} \int_a^\infty (1+|x|)|q_j(x) - q(x)|\,dx = 0, \tag{2.8.12}$$

then

$$\lim_{j\to\infty} \sup_{\rho\in\overline{\Omega_+}} \sup_{x\ge a} |(e_j^{(\nu)}(x,\rho) - e^{(\nu)}(x,\rho))\exp(-i\rho x)| = 0, \quad \nu = 0,1. \tag{2.8.13}$$

If $(1+|x|)|q(x)| \in L(-\infty,a)$, $a < \infty$, *and*

$$\lim_{j\to\infty} \int_{-\infty}^a (1+|x|)|q_j(x) - q(x)|\,dx = 0,$$

then

$$\lim_{j\to\infty} \sup_{\rho\in\overline{\Omega_+}} \sup_{x\le a} |(g_j^{(\nu)}(x,\rho) - g^{(\nu)}(x,\rho))\exp(i\rho x)| = 0, \quad \nu = 0,1. \tag{2.8.14}$$

Here $e_j(x,\rho)$ *and* $g_j(x,\rho)$ *are the Jost solutions for the potentials* q_j.

Proof. Denote

$$z_j(x,\rho) = e_j(x,\rho)\exp(-i\rho x), \quad z(x,\rho) = e(x,\rho)\exp(-i\rho x),$$

$$u_j(x,\rho) = |z_j(x,\rho) - z(x,\rho)|.$$

Then, it follows from (2.7.37) that

$$z_j(x,\rho) - z(x,\rho) = \frac{1}{2i\rho}\int_x^\infty (1-\exp(2i\rho(t-x)))(q(t)z(t,\rho) - q_j(t)z_j(t,\rho))\,dt.$$

From this, taking (2.7.58) into account, we infer

$$u_j(x,\rho) \le \int_x^\infty (t-x)|(q(t)-q_j(t))z(t,\rho)|\,dt + \int_x^\infty (t-x)|q_j(t)|u_j(t,\rho)\,dt.$$

According to (2.8.4),

$$|z(x,\rho)| \le \exp(Q_1^+(x)) \le \exp(Q_1^+(a)), \quad x \ge a, \tag{2.8.15}$$

and consequently

$$u_j(x,\rho) \le \exp(Q_1^+(a))\int_a^\infty (t-a)|q(t)-q_j(t)|\,dt + \int_x^\infty (t-x)|q_j(t)|u_j(t,\rho)\,dt.$$

By virtue of Lemma 2.7.6 this yields

$$u_j(x,\rho) \le \exp(Q_1^+(a)) \int_a^\infty (t-a)|q(t)-q_j(t)|\,dt \exp\Big(\int_x^\infty (t-x)|q_j(t)|\,dt\Big)$$

$$\le \exp(Q_1^+(a)) \int_a^\infty (t-a)|q(t)-q_j(t)|\,dt \exp\Big(\int_a^\infty (t-a)|q_j(t)|\,dt\Big).$$

Hence

$$u_j(x,\rho) \le C_a \int_a^\infty (t-a)|q(t)-q_j(t)|\,dt. \tag{2.8.16}$$

In particular, (2.8.16) and (2.8.12) imply

$$\lim_{j\to\infty} \sup_{\rho\in\overline{\Omega_+}} \sup_{x\ge a} u_j(x,\rho) = 0,$$

and we arrive at (2.8.13) for $\nu = 0$.

Denote

$$v_j(x,\rho) = |(e'_j(x,\rho) - e'(x,\rho))\exp(-i\rho x)|.$$

It follows from (2.7.49) that

$$v_j(x,\rho) \le \int_x^\infty |q(t)z(t,\rho) - q_j(t)z_j(t,\rho)|\,dt,$$

and consequently,

$$v_j(x,\rho) \le \int_a^\infty |(q(t)-q_j(t))z(t,\rho)|\,dt + \int_a^\infty |q_j(t)|u_j(t,\rho)\,dt. \tag{2.8.17}$$

By virtue of (2.8.15)-(2.8.17) we obtain

$$v_j(x,\rho) \le C_a\Big(\int_a^\infty |q(t)-q_j(t)|\,dt + \int_a^\infty (t-a)|q(t)-q_j(t)|\,dt \cdot \int_a^\infty |q_j(t)|\,dt\Big).$$

Together with (2.8.12) this yields

$$\lim_{j\to\infty} \sup_{\rho\in\overline{\Omega_+}} \sup_{x\ge a} v_j(x,\rho) = 0,$$

and we arrive at (2.8.13) for $\nu = 1$. The relations (2.8.14) is proved analogously. □

For real $\rho \ne 0$, the functions $\{e(x,\rho), e(x,-\rho)\}$ and $\{g(x,\rho), g(x,-\rho)\}$ form fundamental systems of solutions for (2.8.1). Therefore, we have for real $\rho \ne 0$:

$$e(x,\rho) = a(\rho)g(x,-\rho) + b(\rho)g(x,\rho),\ g(x,\rho) = c(\rho)e(x,\rho) + d(\rho)e(x,-\rho). \tag{2.8.18}$$

Let us study the properties of the coefficients $a(\rho)$, $b(\rho)$, $c(\rho)$ and $d(\rho)$.

Lemma 2.8.2. *For real* $\rho \ne 0$, *the following relations hold*

$$c(\rho) = -b(-\rho), \quad d(\rho) = a(\rho), \tag{2.8.19}$$

$$\overline{a(\rho)} = a(-\rho), \quad \overline{b(\rho)} = b(-\rho), \tag{2.8.20}$$

$$|a(\rho)|^2 = 1 + |b(\rho)|^2, \tag{2.8.21}$$

$$a(\rho) = -\frac{1}{2i\rho}\langle e(x,\rho), g(x,\rho)\rangle, \quad b(\rho) = \frac{1}{2i\rho}\langle e(x,\rho), g(x,-\rho)\rangle. \tag{2.8.22}$$

Proof. Since $\overline{e(x,\rho)} = e(x,-\rho)$, $\overline{g(x,\rho)} = g(x,-\rho)$, then (2.8.20) follows from (2.8.18). Using (2.8.18) we also calculate

$$\langle e(x,\rho), g(x,\rho)\rangle = \langle a(\rho)g(x,-\rho) + b(\rho)g(x,\rho), g(x,\rho)\rangle = -2i\rho a(\rho),$$

$$\langle e(x,\rho), g(x,-\rho)\rangle = \langle a(\rho)g(x,-\rho) + b(\rho)g(x,\rho), g(x,-\rho)\rangle = 2i\rho b(\rho),$$

$$\langle e(x,\rho), g(x,\rho)\rangle = \langle e(x,\rho), c(\rho)e(x,\rho) + d(\rho)e(x,-\rho)\rangle = 2i\rho d(\rho),$$

$$\langle e(x,-\rho), g(x,\rho)\rangle = e(x,-\rho), c(\rho)e(x,\rho) + d(\rho)e(x,-\rho)\rangle = 2i\rho c(\rho),$$

i.e. (2.8.19) and (2.8.22) are valid. Furthermore,

$$-2i\rho = \langle e(x,\rho), e(x,-\rho)\rangle$$

$$= \langle a(\rho)g(x,-\rho) + b(\rho)g(x,\rho), a(-\rho)g(x,\rho) + b(-\rho)g(x,-\rho)\rangle$$

$$= a(\rho)a(-\rho)\langle g(x,-\rho), g(x,\rho)\rangle + b(\rho)b(-\rho)\langle g(x,\rho), g(x,-\rho)\rangle$$

$$= -2i\rho\Big(|a(\rho)|^2 - |b(\rho)|^2\Big),$$

and we arrive at (2.8.21). □

We note that (2.8.22) gives the analytic continuation for $a(\rho)$ to Ω_+. Hence, the function $a(\rho)$ is analytic in Ω_+, and $\rho a(\rho)$ is continuous in $\overline{\Omega}_+$. The function $\rho b(\rho)$ is continuous for real ρ. Moreover, it follows from (2.8.22) and (2.8.6) that

$$a(\rho) = 1 - \frac{1}{2i\rho}\int_{-\infty}^{\infty} q(t)\,dt + o\left(\frac{1}{\rho}\right),\ b(\rho) = o\left(\frac{1}{\rho}\right),\ |\rho| \to \infty \tag{2.8.23}$$

(in the domains of definition), and consequently the function $\rho(a(\rho) - 1)$ is bounded in $\overline{\Omega}_+$.

Using (2.8.22) and (2.8.8) one can calculate more precisely

$$\left.\begin{aligned} a(\rho) &= 1 - \frac{1}{2i\rho}\int_{-\infty}^{\infty} q(t)\,dt + \frac{1}{2i\rho}\int_0^{\infty} A(t)\exp(i\rho t)\,dt, \\ b(\rho) &= \frac{1}{2i\rho}\int_{-\infty}^{\infty} B(t)\exp(i\rho t)\,dt, \end{aligned}\right\} \tag{2.8.24}$$

where $A(t) \in L(0,\infty)$ and $B(t) \in L(-\infty,\infty)$ are real functions.

Indeed,

$$2i\rho a(\rho) = g(0,\rho)e'(0,\rho) - e(0,\rho)g'(0,\rho)$$

$$= \Big(1 + \int_{-\infty}^{0} A^-(0,t)\exp(-i\rho t)\,dt\Big)\Big(i\rho - A^+(0,0) + \int_0^{\infty} A_1^+(0,t)\exp(i\rho t)\,dt\Big)$$

$$+ \Big(1 + \int_0^{\infty} A^+(0,t)\exp(i\rho t)\,dt\Big)\Big(i\rho - A^-(0,0) - \int_{-\infty}^{0} A_1^-(0,t)\exp(-i\rho t)\,dt\Big).$$

Integration by parts yields

$$i\rho\int_{-\infty}^{0}A^{-}(0,t)\exp(-i\rho t)\,dt=-A^{-}(0,0)+\int_{-\infty}^{0}A_{2}^{-}(0,t)\exp(-i\rho t)\,dt,$$

$$i\rho\int_{0}^{\infty}A^{+}(0,t)\exp(i\rho t)\,dt=-A^{+}(0,0)-\int_{0}^{\infty}A_{2}^{+}(0,t)\exp(i\rho t)\,dt.$$

Furthermore,

$$\int_{-\infty}^{0}A^{-}(0,t)\exp(-i\rho t)\,dt\int_{0}^{\infty}A_{1}^{+}(0,s)\exp(i\rho s)\,ds$$

$$=\int_{-\infty}^{0}A^{-}(0,t)\Big(\int_{-t}^{\infty}A_{1}^{+}(0,\xi+t)\exp(i\rho\xi)\,d\xi\Big)\,dt$$

$$=\int_{0}^{\infty}\Big(\int_{-\xi}^{0}A^{-}(0,t)A_{1}^{+}(0,\xi+t)\,dt\Big)\exp(i\rho\xi)\,d\xi.$$

Analogously,

$$\int_{-\infty}^{0}A_{1}^{-}(0,t)\exp(-i\rho t)\,dt\int_{0}^{\infty}A^{+}(0,s)\exp(i\rho s)\,ds$$

$$=\int_{0}^{\infty}\Big(\int_{-\xi}^{0}A_{1}^{-}(0,t)A^{+}(0,\xi+t)\,dt\Big)\exp(i\rho\xi)\,d\xi.$$

Since

$$2(A^{+}(0,0)+A^{-}(0,0))=\int_{-\infty}^{\infty}q(t)\,dt,$$

we arrive at (2.8.24) for $a(\rho)$, where

$$A(t)=A_{1}^{+}(0,t)-A_{1}^{-}(0,-t)+A_{2}^{-}(0,-t)-A_{2}^{+}(0,t)$$

$$-A^{+}(0,0)A^{-}(0,-t)-A^{-}(0,0)A^{+}(0,t)$$

$$+\int_{-t}^{0}A^{-}(0,\xi)A_{1}^{+}(0,\xi+t)\,d\xi-\int_{-t}^{0}A_{1}^{-}(0,\xi)A^{+}(0,\xi+t)\,d\xi.$$

It follows from (2.8.10)-(2.8.11) that $A(t)\in L(0,\infty)$. For the function $b(\rho)$ the arguments are similar.

Denote

$$e_{0}(x,\rho)=\frac{e(x,\rho)}{a(\rho)},\quad g_{0}(x,\rho)=\frac{g(x,\rho)}{a(\rho)},\tag{2.8.25}$$

$$s^{+}(\rho)=-\frac{b(-\rho)}{a(\rho)},\quad s^{-}(\rho)=\frac{b(\rho)}{a(\rho)}.\tag{2.8.26}$$

The functions $s^{+}(\rho)$ and $s^{-}(\rho)$ are called the *reflection coefficients* (right and left, respectively). It follows from (2.8.18), (2.8.25) and (2.8.26) that

$$e_{0}(x,\rho)=g(x,-\rho)+s^{-}(\rho)g(x,\rho),\quad g_{0}(x,\rho)=e(x,-\rho)+s^{+}(\rho)e(x,\rho).\tag{2.8.27}$$

Using (2.8.25), (2.8.27) and (2.8.3) we get

$$e_{0}(x,\rho)\sim\exp(i\rho x)+s^{-}(\rho)\exp(-i\rho x)\ (x\to-\infty),$$

$$e_0(x,\rho) \sim t(\rho)\exp(i\rho x)\ (x \to \infty),$$

$$g_0(x,\rho) \sim t(\rho)\exp(i\rho x)\ (x \to -\infty),$$

$$g_0(x,\rho) \sim \exp(-i\rho x) + s^+(\rho)\exp(i\rho x)\ (x \to \infty),$$

where $t(\rho) = (a(\rho))^{-1}$ is called the *transmission coefficient*.

We point out *the main properties of the functions* $s^{\pm}(\rho)$. By virtue of (2.8.20)-(2.8.22) and (2.8.26), the functions $s^{\pm}(\rho)$ are continuous for real $\rho \neq 0$, and

$$\overline{s^{\pm}(\rho)} = s^{\pm}(-\rho).$$

Moreover, (2.8.21) implies

$$|s^{\pm}(\rho)|^2 = 1 - \frac{1}{|a(\rho)|^2},$$

and consequently,

$$|s^{\pm}(\rho)| < 1 \ \text{ for real } \ \rho \neq 0.$$

Furthermore, according to (2.8.23) and (2.8.26),

$$s^{\pm}(\rho) = o\left(\frac{1}{\rho}\right) \ \text{ as } \ |\rho| \to \infty.$$

Denote by $R^{\pm}(x)$ the Fourier transform for $s^{\pm}(\rho)$:

$$R^{\pm}(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} s^{\pm}(\rho)\exp(\pm i\rho x)\,d\rho. \tag{2.8.28}$$

Then $R^{\pm}(x) \in L_2(-\infty,\infty)$ are real, and

$$s^{\pm}(\rho) = \int_{-\infty}^{\infty} R^{\pm}(x)\exp(\mp i\rho x)\,dx. \tag{2.8.29}$$

It follows from (2.8.25) and (2.8.27) that

$$\rho e(x,\rho) = \rho a(\rho)\Big((s^-(\rho)+1)g(x,\rho) + g(x,-\rho) - g(x,\rho)\Big),$$

$$\rho g(x,\rho) = \rho a(\rho)\Big((s^+(\rho)+1)e(x,\rho) + e(x,-\rho) - e(x,\rho)\Big),$$

and consequently,

$$\lim_{\rho\to 0} \rho a(\rho)(s^{\pm}(\rho)+1) = 0.$$

Let us now study the properties of the discrete spectrum.

Definition 2.8.1. The values of the parameter λ, for which equation (2.8.1) has nonzero solutions $y(x) \in L_2(-\infty,\infty)$, are called *eigenvalues* of (2.8.1), and the corresponding solutions are called *eigenfunctions*.

The properties of the eigenvalues are similar to the properties of the discrete spectrum of the Sturm-Liouville operator on the half-line (see Section 2.7).

Theorem 2.8.2. *There are no eigenvalues for* $\lambda \geq 0$.

Proof. Repeat the arguments in the proof of Theorems 2.7.12 and 2.7.13. □

Let $\Lambda_+ := \{\lambda,\ \lambda = \rho^2,\ \rho \in \Omega_+ :\ a(\rho) = 0\}$ be the set of zeros of $a(\rho)$ in the upper half-plane Ω_+. Since the function $a(\rho)$ is analytic in Ω_+ and, by virtue of (2.8.23),

$$a(\rho) = 1 + O\Big(\frac{1}{\rho}\Big), \quad |\rho| \to \infty,\ Im\,\rho \geq 0,$$

we get that Λ_+ is an at most countable bounded set.

Theorem 2.8.3. *The set of eigenvalues coincides with* Λ_+. *The eigenvalues* $\{\lambda_k\}$ *are real and negative (i.e.* $\Lambda_+ \subset (-\infty, 0)$*). For each eigenvalue* $\lambda_k = \rho_k^2$, *there exists only one (up to a multiplicative constant) eigenfunction, namely*

$$g(x,\rho_k) = d_k e(x,\rho_k), \quad d_k \neq 0. \tag{2.8.30}$$

The eigenfunctions $e(x,\rho_k)$ *and* $g(x,\rho_k)$ *are real. Eigenfunctions related to different eigenvalues are orthogonal in* $L_2(-\infty,\infty)$.

Proof. Let $\lambda_k = \rho_k^2 \in \Lambda_+$. By virtue of (2.8.22),

$$\langle e(x,\rho_k), g(x,\rho_k)\rangle = 0, \tag{2.8.31}$$

i.e. (2.8.30) is valid. According to Theorem 2.8.1, $e(x,\rho_k) \in L_2(\alpha,\infty)$, $g(x,\rho_k) \in L_2(-\infty,\alpha)$ for each real α. Therefore, (2.8.30) implies

$$e(x,\rho_k),\ g(x,\rho_k) \in L_2(-\infty,\infty).$$

Thus, $e(x,\rho_k)$ and $g(x,\rho_k)$ are eigenfunctions, and $\lambda_k = \rho_k^2$ is an eigenvalue.

Conversely, let $\lambda_k = \rho_k^2$, $\rho_k \in \Omega_+$ be an eigenvalue, and let $y_k(x)$ be a corresponding eigenfunction. Since $y_k(x) \in L_2(-\infty,\infty)$, we have

$$y_k(x) = c_{k1} e(x,\rho_k),\ y_k(x) = c_{k2} g(x,\rho_k),\ c_{k1}, c_{k2} \neq 0,$$

and consequently, (2.8.31) holds. Using (2.8.22) we obtain $a(\rho_k) = 0$, i.e. $\lambda_k \in \Lambda_+$.

Let λ_n and λ_k ($\lambda_n \neq \lambda_k$) be eigenvalues with eigenfunctions $y_n(x) = e(x,\rho_n)$ and $y_k(x) = e(x,\rho_k)$ respectively. Then integration by parts yields

$$\int_{-\infty}^{\infty} \ell y_n(x)\, y_k(x)\, dx = \int_{-\infty}^{\infty} y_n(x)\, \ell y_k(x)\, dx,$$

and hence

$$\lambda_n \int_{-\infty}^{\infty} y_n(x) y_k(x)\, dx = \lambda_k \int_{-\infty}^{\infty} y_n(x) y_k(x)\, dx$$

or

$$\int_{-\infty}^{\infty} y_n(x) y_k(x)\, dx = 0.$$

Furthermore, let $\lambda^0 = u+iv, v\neq 0$ be a non-real eigenvalue with an eigenfunction $y^0(x)\neq 0$. Since $q(x)$ is real, we get that $\overline{\lambda^0}=u-iv$ is also the eigenvalue with the eigenfunction $\overline{y^0(x)}$. Since $\lambda^0\neq\overline{\lambda^0}$, we derive as before

$$\|y^0\|_{L_2}^2=\int_{-\infty}^{\infty} y^0(x)\overline{y^0(x)}\,dx=0,$$

which is impossible. Thus, all eigenvalues $\{\lambda_k\}$ are real, and consequently the eigenfunctions $e(x,\rho_k)$ and $g(x,\rho_k)$ are real too. Together with Theorem 2.8.2 this yields $\Lambda_+\subset(-\infty,0)$. Theorem 2.8.3 is proved. □

For $\lambda_k=\rho_k^2\in\Lambda_+$ we denote

$$\alpha_k^+=\Big(\int_{-\infty}^{\infty} e^2(x,\rho_k)\,dx\Big)^{-1},\quad \alpha_k^-=\Big(\int_{-\infty}^{\infty} g^2(x,\rho_k)\,dx\Big)^{-1}.$$

Theorem 2.8.4. *Λ_+ is a finite set, i.e. in Ω_+ the function $a(\rho)$ has at most a finite number of zeros. All zeros of $a(\rho)$ in Ω_+ are simple, i.e. $a_1(\rho_k)\neq 0$, where $a_1(\rho):=\frac{d}{d\rho}a(\rho)$. Moreover,*

$$\alpha_k^+=\frac{d_k}{ia_1(\rho_k)},\quad \alpha_k^-=\frac{1}{id_k a_1(\rho_k)},\tag{2.8.32}$$

where the numbers d_k are defined by (2.8.30).

Proof. 1) Let us show that

$$\left.\begin{aligned} -2\rho\int_{-A}^{x} e(t,\rho)g(t,\rho)\,dt&=\langle e(t,\rho),\dot g(t,\rho)\rangle\Big|_{-A}^{x},\\ 2\rho\int_{x}^{A} e(t,\rho)g(t,\rho)\,dt&=\langle \dot e(t,\rho),g(t,\rho)\rangle\Big|_{x}^{A},\end{aligned}\right\}\tag{2.8.33}$$

where in this subsection

$$\dot e(t,\rho):=\frac{d}{d\rho}e(t,\rho),\quad \dot g(t,\rho):=\frac{d}{d\rho}g(t,\rho).$$

Indeed,

$$\frac{d}{dx}\langle e(x,\rho),\dot g(x,\rho)\rangle=e(x,\rho)\dot g''(x,\rho)-e''(x,\rho)\dot g(x,\rho).$$

Since

$$-e''(x,\rho)+q(x)e(x,\rho)=\rho^2 e(x,\rho),$$

$$-\dot g''(x,\rho)+q(x)\dot g(x,\rho)=\rho^2\dot g(x,\rho)+2\rho g(x,\rho),$$

we get

$$\frac{d}{dx}\langle e(x,\rho),\dot g(x,\rho)\rangle=-2\rho e(x,\rho)g(x,\rho).$$

Similarly,

$$\frac{d}{dx}\langle \dot e(x,\rho),g(x,\rho)\rangle=2\rho e(x,\rho)g(x,\rho),$$

and we arrive at (2.8.33).

It follows from (2.8.33) that

$$2\rho\int_{-A}^{A} e(t,\rho)g(t,\rho)\,dt = -\langle \dot e(x,\rho), g(x,\rho)\rangle - \langle e(x,\rho), \dot g(x,\rho)\rangle$$

$$+\langle \dot e(x,\rho), g(x,\rho)\rangle\Big|_{x=A} + \langle e(x,\rho), \dot g(x,\rho)\rangle\Big|_{x=-A}.$$

On the other hand, differentiating (2.8.22) with respect to ρ, we obtain

$$2i\rho a_1(\rho) + 2ia(\rho) = -\langle \dot e(x,\rho), g(x,\rho)\rangle - \langle e(x,\rho), \dot g(x,\rho)\rangle.$$

For $\rho = \rho_k$ this yields with the preceding formula

$$ia_1(\rho_k) = \int_{-A}^{A} e(t,\rho_k)g(t,\rho_k)\,dt + \delta_k(A), \tag{2.8.34}$$

where

$$\delta_k(A) = -\frac{1}{2\rho_k}\Big(\langle \dot e(x,\rho_k), g(x,\rho_k)\rangle\Big|_{x=A} + \langle e(x,\rho_k), \dot g(x,\rho_k)\rangle\Big|_{x=-A}\Big).$$

Since $\rho_k = i\tau_k,\ \tau_k > 0,$ we have by virtue of (2.8.4),

$$e(x,\rho_k), e'(x,\rho_k) = O(\exp(-\tau_k x)), \quad x \to +\infty.$$

According to (2.8.8),

$$\dot e(x,\rho_k) = ix\exp(-\tau_k x) + \int_x^{\infty} itA^+(x,t)\exp(-\tau_k t)\,dt,$$

$$\dot e'(x,\rho_k) = i\exp(-\tau_k x) - ix\tau_k\exp(-\tau_k x) - ixA^+(x,x)\exp(-\tau_k x)$$

$$+\int_x^{\infty} itA_1^+(x,t)\exp(-\tau_k t)\,dt.$$

Hence

$$\dot e(x,\rho_k), \dot e'(x,\rho_k) = O(1), \quad x \to +\infty.$$

From this, using (2.8.30), we calculate

$$\langle \dot e(x,\rho_k), g(x,\rho_k)\rangle = d_k\langle \dot e(x,\rho_k), e(x,\rho_k)\rangle = o(1) \text{ as } x \to +\infty,$$

$$\langle e(x,\rho_k), \dot g(x,\rho_k)\rangle = \frac{1}{d_k}\langle g(x,\rho_k), \dot g(x,\rho_k)\rangle = o(1) \text{ as } x \to -\infty.$$

Consequently,

$$\lim_{A\to +\infty} \delta_k(A) = 0.$$

Then (2.8.34) implies

$$ia_1(\rho_k) = \int_{-\infty}^{\infty} e(t,\rho_k)g(t,\rho_k)\,dt.$$

Using (2.8.30) again we obtain

$$ia_1(\rho_k) = d_k \int_{-\infty}^{\infty} e^2(t,\rho_k)\,dt = \frac{1}{d_k}\int_{-\infty}^{\infty} g^2(t,\rho_k)\,dt.$$

Hence $a_1(\rho_k) \neq 0,$ and (2.8.32) is valid.

2) Suppose that $\Lambda_+ = \{\lambda_k\}$ is an infinite set. Since Λ_+ is bounded and $\lambda_k = \rho_k^2 < 0,$ it follows that $\rho_k = i\tau_k \to 0,\ \tau_k > 0.$ By virtue of (2.8.4)-(2.8.5), there exists $A > 0$ such that

$$\left.\begin{aligned} e(x,i\tau) &\geq \tfrac{1}{2}\exp(-\tau x) \ \text{ for } x \geq A,\ \tau \geq 0, \\ g(x,i\tau) &\geq \tfrac{1}{2}\exp(\tau x) \ \text{ for } x \leq -A,\ \tau \geq 0, \end{aligned}\right\} \tag{2.8.35}$$

and consequently

$$\left.\begin{aligned} \int_{A}^{\infty} e(x,\rho_k)e(x,\rho_n)\,dx &\geq \frac{\exp(-(\tau_k+\tau_n)A)}{4(\tau_k+\tau_n)} \geq \frac{\exp(-2AT)}{8T}, \\ \int_{-\infty}^{-A} g(x,\rho_k)g(x,\rho_n)\,dx &\geq \frac{\exp(-(\tau_k+\tau_n)A)}{4(\tau_k+\tau_n)} \geq \frac{\exp(-2AT)}{8T}, \end{aligned}\right\} \tag{2.8.36}$$

where $T = \max\limits_k \tau_k.$ Since the eigenfunctions $e(x,\rho_k)$ and $e(x,\rho_n)$ are orthogonal in $L_2(-\infty,\infty)$ we get

$$\begin{aligned} 0 &= \int_{-\infty}^{\infty} e(x,\rho_k)e(x,\rho_n)\,dx \\ &= \int_{A}^{\infty} e(x,\rho_k)e(x,\rho_n)\,dx + \frac{1}{d_kd_n}\int_{-\infty}^{-A} g(x,\rho_k)g(x,\rho_n)\,dx \\ &+ \int_{-A}^{A} e^2(x,\rho_k)\,dx + \int_{-A}^{A} e(x,\rho_k)(e(x,\rho_n)-e(x,\rho_k))\,dx. \end{aligned} \tag{2.8.37}$$

Take $x_0 \leq -A$ such that $e(x_0,0) \neq 0.$ According to (2.8.30),

$$\frac{1}{d_kd_n} = \frac{e(x_0,\rho_k)e(x_0,\rho_n)}{g(x_0,\rho_k)g(x_0,\rho_n)}.$$

Since the functions $e(x,\rho)$ and $g(x,\rho)$ are continuous for $Im\,\rho \geq 0,$ we calculate with the help of (2.8.35),

$$\lim_{k,n\to\infty} g(x_0,\rho_k)g(x_0,\rho_n) = g^2(x_0,0) > 0,$$

$$\lim_{k,n\to\infty} e(x_0,\rho_k)e(x_0,\rho_n) = e^2(x_0,0) > 0.$$

Therefore,

$$\lim_{k,n\to\infty} \frac{1}{d_kd_n} > 0.$$

Together with (2.8.36) this yields

$$\int_{A}^{\infty} e(x,\rho_k)e(x,\rho_n)\,dx + \frac{1}{d_kd_n}\int_{-\infty}^{-A} g(x,\rho_k)g(x,\rho_n)\,dx$$

$$+\int_{-A}^{A} e^2(x,\rho_k)\,dx \ge C > 0 \tag{2.8.38}$$

for sufficiently large k and n. On the other hand, by standard technique [27, Ch.3] one can easily verify that

$$\int_{-A}^{A} e(x,\rho_k)(e(x,\rho_n)-e(x,\rho_k))\,dx \to 0 \ \text{ as } \ k,n\to\infty. \tag{2.8.39}$$

The relations (2.8.37)-(2.8.39) give us a contradiction.
This means that Λ_+ is a finite set. □

Thus, the set of eigenvalues has the form

$$\Lambda_+ = \{\lambda_k\}_{k=\overline{1,N}}, \quad \lambda_k = \rho_k^2,\ \rho_k = i\tau_k,\ 0<\tau_1<\ldots<\tau_m.$$

Definition 2.8.2. The data $J^+ = \{s^+(\rho),\lambda_k,\alpha_k^+;\ \rho\in\mathbf{R},\ k=\overline{1,N}\}$ are called the right scattering data, and the data $J^- = \{s^-(\rho),\lambda_k,\alpha_k^-;\ \rho\in\mathbf{R},\ k=\overline{1,N}\}$ are called the left scattering data.

Example 2.8.1. Let $q(x)\equiv 0$. Then

$$e(x,\rho)=\exp(i\rho x), \quad g(x,\rho)=\exp(-i\rho x),$$

$$a(\rho)=1, \quad b(\rho)=0, \quad s^{\pm}(\rho)=0, \quad N=0,$$

i.e. there are no eigenvalues at all.

Let us study connections between the scattering data J^+ and J^-. Consider the function

$$\gamma(\rho)=\frac{1}{a(\rho)}\prod_{k=1}^{N}\frac{\rho-i\tau_k}{\rho+i\tau_k}. \tag{2.8.40}$$

Lemma 2.8.3. (i_1) *The function* $\gamma(\rho)$ *is analytic in* Ω_+ *and continuous in* $\overline{\Omega_+}\setminus\{0\}$.
(i_2) $\gamma(\rho)$ *has no zeros in* $\overline{\Omega_+}\setminus\{0\}$.
(i_3) *For* $|\rho|\to\infty,\ \rho\in\overline{\Omega_+}$,

$$\gamma(\rho)=1+O\Big(\frac{1}{\rho}\Big). \tag{2.8.41}$$

(i_4) $|\gamma(\rho)|\le 1$ *for* $\rho\in\overline{\Omega_+}$.

Proof. The assertions $(i_1)-(i_3)$ are obvious consequences of the preceding discussion, and only (i_4) needs to be proved. By virtue of (2.8.21), $|a(\rho)|\ge 1$ for real $\rho\ne 0$, and consequently,

$$|\gamma(\rho)|\le 1 \ \text{ for real } \ \rho\ne 0. \tag{2.8.42}$$

Suppose that the function $\rho a(\rho)$ is analytic in the origin. Then, using (2.8.40) and (2.8.42) we deduce that the function $\gamma(\rho)$ has a removable singularity in the origin, and $\gamma(\rho)$ (after extending continuously to the origin) is continuous in $\overline{\Omega_+}$. Using (2.8.41), (2.8.42) and the maximum principle we arrive at (i_4).

In the general case we cannot use these arguments for $\gamma(\rho)$. Therefore, we introduce the potentials

$$q_r(x) = \begin{cases} q(x), & |x| \le r, \\ 0, & |x| > r, \end{cases} \qquad r \ge 0,$$

and consider the corresponding Jost solutions $e_r(x,\rho)$ and $g_r(x,\rho)$. Clearly, $e_r(x,\rho) \equiv \exp(i\rho x)$ for $x \ge r$ and $g_r(x,\rho) \equiv \exp(-i\rho x)$ for $x \le -r$. For each fixed x, the functions $e_r^{(\nu)}(x,\rho)$ and $g_r^{(\nu)}(x,\rho)$ $(\nu = 0,1)$ are entire in ρ. Take

$$a_r(\rho) = -\frac{1}{2i\rho}\langle e_r(x,\rho), g_r(x,\rho)\rangle, \quad \gamma_r(\rho) = \frac{1}{a_r(\rho)} \prod_{k=1}^{N_r} \frac{\rho - i\tau_{kr}}{\rho + i\tau_{kr}},$$

where $\rho_{kr} = i\tau_{kr}$, $k = \overline{1,N_r}$ are zeros of $a_r(\rho)$ in the upper half-plane Ω_+. The function $\rho a_r(\rho)$ is entire in ρ, and (see Lemma 2.8.2) $|a_r(\rho)| \ge 1$ for real ρ. The function $\gamma_r(\rho)$ is analytic in $\overline{\Omega_+}$, and

$$|\gamma_r(\rho)| \le 1 \text{ for } \rho \in \overline{\Omega_+}. \tag{2.8.43}$$

By virtue of Lemma 2.8.1,

$$\lim_{r\to\infty} \sup_{\rho\in\overline{\Omega_+}} \sup_{x\ge a} |(e_r^{(\nu)}(x,\rho) - e^{(\nu)}(x,\rho))\exp(-i\rho x)| = 0,$$

$$\lim_{r\to\infty} \sup_{\rho\in\overline{\Omega_+}} \sup_{x\le a} |(g_r^{(\nu)}(x,\rho) - g^{(\nu)}(x,\rho))\exp(i\rho x)| = 0,$$

for $\nu = 0,1$ and each real a. Therefore

$$\lim_{r\to\infty} \sup_{\rho\in\overline{\Omega_+}} |\rho(a_r(\rho) - a(\rho))| = 0,$$

i.e.

$$\lim_{r\to\infty} \rho a_r(\rho) = \rho a(\rho) \text{ uniformly in } \overline{\Omega_+}. \tag{2.8.44}$$

In particular, (2.8.44) yields that $0 < \tau_{kr} \le C$ for all k and r.

Let δ_r be the infimum of distances between the zeros $\{\rho_{kr}\}$ of $a_r(\rho)$ in the upper half-plane $Im\,\rho > 0$. Let us show that

$$\delta^* := \inf_{r>0} \delta_r > 0. \tag{2.8.45}$$

Indeed, suppose on the contrary that there exists a sequence $r_k \to \infty$ such that $\delta_{r_k} \to 0$. Let $\rho_k^{(1)} = i\tau_k^{(1)}$, $\rho_k^{(2)} = i\tau_k^{(2)}$ $(\tau_k^{(1)}, \tau_k^{(2)} \ge 0)$ be zeros of $a_{r_k}(\rho)$ such that $\rho_k^{(1)} - \rho_k^{(2)} \to 0$ as $k \to \infty$. It follows from (2.8.4)-(2.8.5) that there exists $A > 0$ such that

$$\left.\begin{aligned} e_r(x,i\tau) &\ge \frac{1}{2}\exp(-\tau x) \text{ for } x \ge A,\ \tau \ge 0,\ r \ge 0, \\ g_r(x,i\tau) &\ge \frac{1}{2}\exp(\tau x) \text{ for } x \le -A,\ \tau \ge 0,\ r \ge 0. \end{aligned}\right\} \tag{2.8.46}$$

Since the functions $e_{r_k}(x,\rho_k^{(1)})$ and $e_{r_k}(x,\rho_k^{(2)})$ are orthogonal in $L_2(-\infty,\infty)$, we get

$$0=\int_{-\infty}^{\infty} e_{r_k}(x,\rho_k^{(1)})e_{r_k}(x,\rho_k^{(2)})\,dx$$

$$=\int_{A}^{\infty} e_{r_k}(x,\rho_k^{(1)})e_{r_k}(x,\rho_k^{(2)})\,dx+\frac{1}{d_k^{(1)}d_k^{(2)}}\int_{-\infty}^{-A} g_{r_k}(x,\rho_k^{(1)})g_{r_k}(x,\rho_k^{(2)})\,dx$$

$$+\int_{-A}^{A} e_{r_k}^2(x,\rho_k^{(1)})\,dx+\int_{-A}^{A} e_{r_k}(x,\rho_k^{(1)})(e_{r_k}(x,\rho_k^{(2)})-e_{r_k}(x,\rho_k^{(1)}))\,dx, \tag{2.8.47}$$

where the numbers $d_k^{(j)}$ are defined by

$$g_{r_k}(x,\rho_k^{(j)})=d_k^{(j)}e_{r_k}(x,\rho_k^{(j)}),\quad d_k^{(j)}\neq 0.$$

Take $x_0\le -A$. Then, by virtue of (2.8.46),

$$g_{r_k}(x_0,\rho_k^{(1)})g_{r_k}(x_0,\rho_k^{(2)})\ge C>0,$$

and

$$\frac{1}{d_k^{(1)}d_k^{(2)}}=\frac{e_{r_k}(x_0,\rho_k^{(1)})e_{r_k}(x_0,\rho_k^{(2)})}{g_{r_k}(x_0,\rho_k^{(1)})g_{r_k}(x_0,\rho_k^{(2)})}.$$

Using Lemma 2.8.1 we get

$$\varliminf_{k\to\infty} e_{r_k}(x_0,\rho_k^{(1)})e_{r_k}(x_0,\rho_k^{(2)})\ge 0;$$

hence

$$\varliminf_{k\to\infty}\frac{1}{d_k^{(1)}d_k^{(2)}}\ge 0.$$

Then

$$\int_{A}^{\infty} e_{r_k}(x,\rho_k^{(1)})e_{r_k}(x,\rho_k^{(2)})\,dx+\frac{1}{d_k^{(1)}d_k^{(2)}}\int_{-\infty}^{-A} g_{r_k}(x,\rho_k^{(1)})g_{r_k}(x,\rho_k^{(2)})\,dx$$

$$+\int_{-A}^{A} e_{r_k}^2(x,\rho_k^{(1)})\,dx\ge C>0.$$

On the other hand,

$$\int_{-A}^{A} e_{r_k}(x,\rho_k^{(1)})(e_{r_k}(x,\rho_k^{(2)})-e_{r_k}(x,\rho_k^{(1)}))\,dx\to 0 \text{ as } k\to\infty.$$

From this and (2.8.47) we arrive at a contradiction, i.e. (2.8.45) is proved.

Let $D_{\delta,R}:=\{\rho\in\Omega_+:\ \delta<|\rho|<R\}$, where $0<\delta<\min(\delta^*,\tau_1)$, $R>\tau_N$. Using (2.8.44) one can show that

$$\lim_{r\to\infty}\gamma_r(\rho)=\gamma(\rho) \text{ uniformly in } \overline{D_{\delta,R}}. \tag{2.8.48}$$

It follows from (2.8.43) and (2.8.48) that $|\gamma(\rho)|\le 1$ for $\rho\in\overline{D_{\delta,R}}$. By virtue of arbitrariness of δ and R we obtain $|\gamma(\rho)|\le 1$ for $\rho\in\overline{\Omega_+}$, i.e. (i_4) is proved. □

It follows from Lemma 2.8.3 that

$$\frac{1}{a(\rho)} = O(1) \ \text{as} \ |\rho| \to 0, \ \rho \in \overline{\Omega_+}. \tag{2.8.49}$$

We also note that since the function $\sigma a(\sigma)$ is continuous at the origin, it follows that for sufficiently small real σ,

$$1 \leq |a(\sigma)| = \frac{1}{|\gamma(\sigma)|} \leq \frac{C}{|\sigma|}.$$

The properties of the function $\gamma(\rho)$ obtained in Lemma 2.8.3 allow one to recover $\gamma(\rho)$ in Ω_+ from its modulus $|\gamma(\sigma)|$ given for real σ.

Lemma 2.8.4. *The following relation holds*

$$\gamma(\rho) = \exp\Big(\frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{\ln|\gamma(\xi)|}{\xi-\rho}\,d\xi\Big), \quad \rho \in \Omega_+. \tag{2.8.50}$$

Proof. 1) The function $\ln\gamma(\rho)$ is analytic in Ω_+ and $\ln\gamma(\rho) = O(\rho^{-1})$ for $|\rho| \to \infty$, $\rho \in \overline{\Omega_+}$. Consider the closed contour C_R (with counterclockwise circuit) which is the boundary of the domain $D_R = \{\rho \in \Omega_+ : \ |\rho| < R\}$ (see fig. 2.8.1). By Cauchy's integral formula [14, p.84],

$$\ln\gamma(\rho) = \frac{1}{2\pi i}\int_{C_R} \frac{\ln\gamma(\xi)}{\xi-\rho}\,d\xi, \quad \rho \in D_R.$$

Since

$$\lim_{R\to\infty} \frac{1}{2\pi i}\int_{\substack{|\xi|=R \\ \xi\in\overline{\Omega_+}}} \frac{\ln\gamma(\xi)}{\xi-\rho}\,d\xi = 0,$$

we obtain

$$\ln\gamma(\rho) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\ln\gamma(\xi)}{\xi-\rho}\,d\xi, \quad \rho \in \Omega_+. \tag{2.8.51}$$

2) Take a real σ and the closed contour $C^{\sigma}_{R,\delta}$ (with counterclockwise circuit) consisting of the semicircles $C^0_R = \{\xi : \ \xi = R\exp(i\varphi), \ \varphi \in [0,\pi]\}$, $\quad \Gamma^{\sigma}_{\delta} = \{\xi : \ \xi - \sigma = \delta\exp(i\varphi), \ \varphi \in [0,\pi]\}$, $\delta > 0$ and the intervals $[-R,R] \setminus [\sigma-\delta, \sigma+\delta]$ (see fig. 2.8.1).

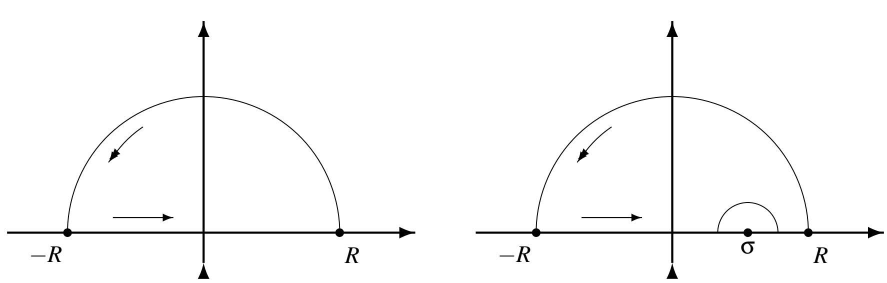

Figure 2.8.1.

By Cauchy's theorem,

$$\frac{1}{2\pi i}\int_{C^{\sigma}_{R,\delta}} \frac{\ln\gamma(\xi)}{\xi-\sigma}\,d\xi = 0.$$

Since

$$\lim_{R\to\infty}\frac{1}{2\pi i}\int_{C^0_R} \frac{\ln\gamma(\xi)}{\xi-\sigma}\,d\xi = 0, \quad \lim_{\delta\to 0}\frac{1}{2\pi i}\int_{\Gamma^{\sigma}_{\delta}} \frac{\ln\gamma(\xi)}{\xi-\sigma}\,d\xi = -\frac{1}{2}\ln\gamma(\sigma),$$

we get for real σ,

$$\ln\gamma(\sigma) = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{\ln\gamma(\xi)}{\xi-\sigma}\,d\xi. \tag{2.8.52}$$

In (2.8.52) (and everywhere below where necessary) the integral is understood in the principal value sense.

3) Let $\gamma(\sigma) = |\gamma(\sigma)|\exp(-i\beta(\sigma))$. Separating in (2.8.52) real and imaginary parts we obtain

$$\beta(\sigma) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\ln|\gamma(\xi)|}{\xi-\sigma}\,d\xi, \quad \ln|\gamma(\sigma))| = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\beta(\xi)}{\xi-\sigma}\,d\xi.$$

Then, using (2.8.51) we calculate for $\rho\in\Omega_+$:

$$\ln\gamma(\rho) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\ln|\gamma(\xi)|}{\xi-\rho}\,d\xi - \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\beta(\xi)}{\xi-\rho}\,d\xi$$

$$= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\ln|\gamma(\xi)|}{\xi-\rho}\,d\xi - \frac{1}{2\pi^2}\int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty} \frac{d\xi}{(\xi-\rho)(s-\xi)}\Big)\ln|\gamma(s)|\,ds.$$

Since

$$\frac{1}{(\xi-\rho)(s-\xi)} = \frac{1}{s-\rho}\Big(\frac{1}{\xi-\rho} - \frac{1}{\xi-s}\Big),$$

it follows that for $\rho\in\Omega_+$ and real s,

$$\int_{-\infty}^{\infty} \frac{d\xi}{(\xi-\rho)(s-\xi)} = \frac{\pi i}{s-\rho}.$$

Consequently,

$$\ln\gamma(\rho) = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{\ln|\gamma(\xi)|}{\xi-\rho}\,d\xi, \quad \rho\in\Omega_+,$$

and we arrive at (2.8.50). □

It follows from (2.8.21) and (2.8.26) that for real $\rho\neq 0$,

$$\frac{1}{|a(\rho)|^2} = 1-|s^{\pm}(\rho)|^2.$$

By virtue of (2.8.40) this yields for real $\rho\neq 0$,

$$|\gamma(\rho)| = \sqrt{1-|s^{\pm}(\rho)|^2}.$$

Using (2.8.40) and (2.8.50) we obtain for $\rho\in\Omega_+$:

$$a(\rho) = \prod_{k=1}^{N}\frac{\rho-i\tau_k}{\rho+i\tau_k}\exp\Big(-\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\ln(1-|s^{\pm}(\xi)|^2)}{\xi-\rho}\,d\xi\Big). \tag{2.8.53}$$

We note that since the function $\rho a(\rho)$ is continuous in $\overline{\Omega_+}$, it follows that

$$\frac{\rho^2}{1-|s^{\pm}(\rho)|^2} = O(1) \text{ as } |\rho| \to 0.$$

Relation (2.8.53) allows one to establish connections between the scattering data J^+ and J^-. More precisely, from the given data J^+ one can uniquely reconstruct J^- (and vice versa) by the following algorithm.

Algorithm 2.8.1. Let J^+ be given. Then
1) construct the function $a(\rho)$ by (2.8.53);
2) calculate d_k and α_k^-, $k=\overline{1,N}$ by (2.8.32);
3) find $b(\rho)$ and $s^-(\rho)$ by (2.8.26).

2. Solution of the inverse scattering problem

The *inverse scattering problem* is formulated as follows: given the scattering data J^+ (or J^-), construct the potential q.

The central role for constructing the solution of the inverse scattering problem is played by the so-called main equation which is a linear integral equation of Fredholm type. We give a derivation of the main equation and study its properties. Using the main equation we provide the solution of the inverse scattering problem along with necessary and sufficient conditions of its solvability.

Theorem 2.8.5. *For each fixed* x, *the functions* $A^{\pm}(x,t)$, *defined in* (2.8.8)*, satisfy the integral equations*

$$F^+(x+y)+A^+(x,y)+\int_x^\infty A^+(x,t)F^+(t+y)\,dt=0, \quad y>x, \tag{2.8.54}$$

$$F^-(x+y)+A^-(x,y)+\int_{-\infty}^x A^-(x,t)F^-(t+y)\,dt=0, \quad y<x, \tag{2.8.55}$$

where

$$F^{\pm}(x)=R^{\pm}(x)+\sum_{k=1}^N \alpha_k^{\pm}\exp(\mp\tau_k x), \tag{2.8.56}$$

and the functions $R^{\pm}(x)$ *are defined by* (2.8.28)*.*

Equations (2.8.54) *and* (2.8.55) *are called the main equations or Gelfand-Levitan-Marchenko equations for the inverse scattering problem.*

Proof. By virtue of (2.8.18) and (2.8.19),

$$\left(\frac{1}{a(\rho)}-1\right)g(x,\rho)=s^+(\rho)e(x,\rho)+e(x,-\rho)-g(x,\rho). \tag{2.8.57}$$

Put $A^+(x,t)=0$ for $t<x$, and $A^-(x,t)=0$ for $t>x$. Then, using (2.8.8) and (2.8.29), we get

$$s^+(\rho)e(x,\rho)+e(x,-\rho)-g(x,\rho)$$

$$= \left(\int_{-\infty}^{\infty} R^+(y)\exp(i\rho y)dy\right)\left(\exp(i\rho x)+\int_{-\infty}^{\infty} A^+(x,t)\exp(i\rho t)\,dt\right)$$

$$+\int_{-\infty}^{\infty}(A^+(x,t)-A^-(x,t))\exp(-i\rho t)\,dt=\int_{-\infty}^{\infty} H(x,y)\exp(-i\rho y)\,dy,$$

where

$$H(x,y)=A^+(x,y)-A^-(x,y)+R^+(x+y)+\int_x^{\infty} A^+(x,t)R^+(t+y)\,dt. \tag{2.8.58}$$

Thus, for each fixed x, the right-hand side in (2.8.57) is the Fourier transform of the function $H(x,y)$. Hence

$$H(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1}{a(\rho)}-1\right)g(x,\rho)\exp(i\rho y)\,d\rho. \tag{2.8.59}$$

Fix x and $y\ (y>x)$ and consider the function

$$f(\rho):=\left(\frac{1}{a(\rho)}-1\right)g(x,\rho)\exp(i\rho y). \tag{2.8.60}$$

According to (2.8.6) and (2.8.23),

$$f(\rho)=\frac{c}{\rho}\exp(i\rho(y-x))(1+o(1)),\quad |\rho|\to\infty,\ \rho\in\overline{\Omega_+}. \tag{2.8.61}$$

Let $C_{\delta,R}$ be a closed contour (with counterclockwise circuit) which is the boundary of the domain $D_{\delta,R}=\{\rho\in\Omega_+:\ \delta<|\rho|<R\}$, where $\delta<\tau_1<\ldots<\tau_N<R$. Thus, all zeros $\rho_k=i\tau_k,\ k=\overline{1,N}$ of $a(\rho)$ are contained in $D_{\delta,R}$. By the residue theorem,

$$\frac{1}{2\pi i}\int_{C_{\delta,R}} f(\rho)\,d\rho=\sum_{k=1}^{N}\operatorname*{Res}_{\rho=\rho_k} f(\rho).$$

On the other hand, it follows from (2.8.60)-(2.8.61), (2.8.5) and (2.8.49) that

$$\lim_{R\to\infty}\frac{1}{2\pi i}\int_{\substack{|\rho|=R\\ \rho\in\overline{\Omega_+}}} f(\rho)\,d\rho=0,\quad \lim_{\delta\to 0}\frac{1}{2\pi i}\int_{\substack{|\rho|=\delta\\ \rho\in\overline{\Omega_+}}} f(\rho)\,d\rho=0.$$

Hence

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} f(\rho)\,d\rho=\sum_{k=1}^{N}\operatorname*{Res}_{\rho=\rho_k} f(\rho).$$

From this and (2.8.59)-(2.8.60) it follows that

$$H(x,y)=i\sum_{k=1}^{N}\frac{g(x,i\tau_k)\exp(-\tau_k y)}{a_1(i\tau_k)}.$$

Using the fact that all eigenvalues are simple, (2.8.30), (2.8.8) and (2.8.32) we obtain

$$H(x,y)=i\sum_{k=1}^{N}\frac{d_k e(x,i\tau_k)\exp(-\tau_k y)}{a_1(i\tau_k)}$$

$$= -\sum_{k=1}^{N} \alpha_k^+ \left(\exp(-\tau_k(x+y)) + \int_x^\infty A^+(x,t) \exp(-\tau_k(t+y))\, dt \right). \tag{2.8.62}$$

Since $A^-(x,y) = 0$ for $y > x$, (2.8.58) and (2.8.62) yield (2.8.54). Relation (2.8.55) is proved analogously. □

Lemma 2.8.5. *Let nonnegative functions* $v(x), u(x)$ $(a \le x \le T \le \infty)$ *be given such that* $v(x) \in L(a,T)$, $u(x)v(x) \in L(a,T)$, *and let* $c_1 \ge 0$. *If*

$$u(x) \le c_1 + \int_x^T v(t)u(t)\, dt, \tag{2.8.63}$$

then

$$u(x) \le c_1 \exp\Big(\int_x^T v(t)\, dt \Big). \tag{2.8.64}$$

Proof. Denote

$$\xi(x) := c_1 + \int_x^T v(t)u(t)\, dt.$$

Then $\xi(T) = c_1$, $-\xi'(x) = v(x)u(x)$, and (2.8.63) yields

$$0 \le -\xi'(x) \le v(x)\xi(x).$$

Let $c_1 > 0$. Then $\xi(x) > 0$, and

$$0 \le -\frac{\xi'(x)}{\xi(x)} \le v(x).$$

Integrating this inequality we obtain

$$\ln \frac{\xi(x)}{\xi(T)} \le \int_x^T v(t)\, dt,$$

and consequently,

$$\xi(x) \le c_1 \exp\Big(\int_x^T v(t)\, dt \Big).$$

According to (2.8.63) $u(x) \le \xi(x)$, and we arrive at (2.8.64).

If $c_1 = 0$, then $\xi(x) = 0$. Indeed, suppose on the contrary that $\xi(x) \ne 0$. Then, there exists $T_0 \le T$ such that $\xi(x) > 0$ for $x < T_0$, and $\xi(x) \equiv 0$ for $x \in [T_0, T]$. Repeating the arguments we get for $x < T_0$ and sufficiently small $\varepsilon > 0$,

$$\ln \frac{\xi(x)}{\xi(T_0 - \varepsilon)} \le \int_x^{T_0-\varepsilon} v(t)\, dt \le \int_x^{T_0} v(t)\, dt,$$

which is impossible. Thus, $\xi(x) \equiv 0$, and (2.8.64) becomes obvious. □

Lemma 2.8.6. *The functions* $F^\pm(x)$ *are absolutely continuous and for each fixed* $a > -\infty$,

$$\int_a^\infty |F^\pm(\pm x)|\, dx < \infty, \quad \int_a^\infty (1+|x|)|F^{\pm\prime}(\pm x)|\, dx < \infty. \tag{2.8.65}$$

Proof. 1) According to (2.8.56) and (2.8.28), $F^+(x)\in L_2(a,\infty)$ for each fixed $a>-\infty$. By continuity, (2.8.54) is also valid for $y=x$:

$$F^+(2x)+A^+(x,x)+\int_x^\infty A^+(x,t)F^+(t+x)\,dt=0. \tag{2.8.66}$$

Rewrite (2.8.66) to the form

$$F^+(2x)+A^+(x,x)+2\int_x^\infty A^+(x,2\xi-x)F^+(2\xi)\,d\xi=0. \tag{2.8.67}$$

It follows from (2.8.67) and (2.8.10) that the function $F^+(x)$ is continuous, and for $x\ge a$,

$$|F^+(2x)|\le\frac{1}{2}Q_0^+(x)+\exp(Q_1^+(a))\int_x^\infty Q_0^+(\xi)|F^+(2\xi)|\,d\xi. \tag{2.8.68}$$

Fix $r\ge a$. Then for $x\ge r$, (2.8.68) yields

$$|F^+(2x)|\le\frac{1}{2}Q_0^+(r)+\exp(Q_1^+(a))\int_x^\infty Q_0^+(\xi)|F^+(2\xi)|\,d\xi.$$

Applying Lemma 2.8.5 we obtain

$$|F^+(2x)|\le\frac{1}{2}Q_0^+(r)\exp(Q_1^+(a)\exp(Q_1^+(a))),\quad x\ge r\ge a,$$

and consequently

$$|F^+(2x)|\le C_aQ_0^+(x),\quad x\ge a. \tag{2.8.69}$$

It follows from (2.8.69) that for each fixed $a>-\infty$,

$$\int_a^\infty |F^+(x)|\,dx<\infty.$$

2) By virtue of (2.8.67), the function $F^+(x)$ is absolutely continuous, and

$$2F^{+\prime}(2x)+\frac{d}{dx}A^+(x,x)-2A^+(x,x)F^+(2x)$$
$$+2\int_x^\infty\Big(A_1^+(x,2\xi-x)+A_2^+(x,2\xi-x)\Big)F^+(2\xi)\,d\xi=0,$$

where

$$A_1^+(x,t)=\frac{\partial A^+(x,t)}{\partial x},\quad A_2^+(x,t)=\frac{\partial A^+(x,t)}{\partial t}.$$

Taking (2.8.9) into account we get

$$F^{+\prime}(2x)=\frac{1}{4}q(x)+P(x), \tag{2.8.70}$$

where

$$P(x)=-\int_x^\infty\Big(A_1^+(x,2\xi-x)+A_2^+(x,2\xi-x)\Big)F^+(2\xi)\,d\xi$$
$$-\frac{1}{2}F^+(2x)\int_x^\infty q(t)\,dt.$$

It follows from (2.8.69) and (2.8.11) that

$$|P(x)| \le C_a (Q_0^+(x))^2, \quad x \ge a. \tag{2.8.71}$$

Since

$$xQ_0^+(x) \le \int_x^\infty t|q(t)|\,dt,$$

it follows from (2.8.70) and (2.8.71) that for each fixed $a > -\infty$,

$$\int_a^\infty (1+|x|)|F^{+\prime}(x)|\,dx < \infty,$$

and (2.8.65) is proved for the function $F^+(x)$. For $F^-(x)$ the arguments are similar. □

Now we are going to study the solvability of the main equations (2.8.54) and (2.8.55). Let sets $J^\pm = \{s^\pm(\rho), \lambda_k, \alpha_k^\pm;\ \rho \in \mathbf{R},\ k = \overline{1,N}\}$ be given satisfying the following condition.

Condition A. *For real $\rho \neq 0$, the functions $s^\pm(\rho)$ are continuous, $|s^\pm(\rho)| < 1$, $\overline{s^\pm(\rho)} = s^\pm(-\rho)$ and $s^\pm(\rho) = o\left(\frac{1}{\rho}\right)$ as $|\rho| \to \infty$. The real functions $R^\pm(x)$, defined by* (2.8.28)*, are absolutely continuous, $R^\pm(x) \in L_2(-\infty,\infty)$, and for each fixed $a > -\infty$,*

$$\int_a^\infty |R^\pm(\pm x)|\,dx < \infty, \quad \int_a^\infty (1+|x|)|R^{\pm\prime}(\pm x)|\,dx < \infty. \tag{2.8.72}$$

Moreover, $\lambda_k = -\tau_k^2 < 0$, $\alpha_k^\pm > 0$, $k = \overline{1,N}$.

Theorem 2.8.6. *Let sets J^+ (J^-) be given satisfying Condition A. Then for each fixed x, the integral equation* (2.8.54)((2.8.55) *respectively*) *has a unique solution $A^+(x,y) \in L(x,\infty)$ ($A^-(x,y) \in L(-\infty,x)$ respectively*).

Proof. For definiteness we consider equation (2.8.54). For (2.8.55) the arguments are the same. It is easy to check that for each fixed x, the operator

$$(J_x f)(y) = \int_x^\infty F^+(t+y) f(t)\,dt, \quad y > x$$

is compact in $L(x,\infty)$. Therefore, it is sufficient to prove that the homogeneous equation

$$f(y) + \int_x^\infty F^+(t+y) f(t)\,dt = 0 \tag{2.8.73}$$

has only the zero solution. Let $f(y) \in L(x,\infty)$ be a real function satisfying (2.8.73). It follows from (2.8.73) and Condition A that the functions $F^+(y)$ and $f(y)$ are bounded on the half-line $y > x$, and consequently $f(y) \in L_2(x,\infty)$. Using (2.8.56) and (2.8.28) we calculate

$$0 = \int_x^\infty f^2(y)\,dy + \int_x^\infty \int_x^\infty F^+(t+y) f(t) f(y)\,dt dy$$

$$= \int_x^\infty f^2(y)\,dy + \sum_{k=1}^N \alpha_k^+ \Big(\int_x^\infty f(y)\exp(-\tau_k y)\,dy\Big)^2 + \frac{1}{2\pi}\int_{-\infty}^\infty s^+(\rho)\Phi^2(\rho)\,d\rho,$$

where

$$\Phi(\rho) = \int_x^\infty f(y)\exp(i\rho y)\,dy.$$

According to Parseval's equality

$$\int_x^\infty f^2(y)\,dy = \frac{1}{2\pi}\int_{-\infty}^\infty |\Phi(\rho)|^2\,d\rho,$$

and hence

$$\sum_{k=1}^N \alpha_k^+ \Big(\int_x^\infty f(y)\exp(-\tau_k y)\,dy\Big)^2$$
$$+\frac{1}{2\pi}\int_{-\infty}^\infty |\Phi(\rho)|^2\Big(1-|s^+(\rho)|\exp(i(2\theta(\rho)+\eta(\rho)))\Big)\,d\rho = 0,$$

where $\theta(\rho) = \arg\Phi(\rho),\ \eta(\rho) = \arg(-s^+(\rho))$. In this equality we take the real part:

$$\sum_{k=1}^N \alpha_k^+ \Big(\int_x^\infty f(y)\exp(-\tau_k y)\,dy\Big)^2$$
$$+\frac{1}{2\pi}\int_{-\infty}^\infty |\Phi(\rho)|^2\Big(1-|s^+(\rho)|\cos((2\theta(\rho)+\eta(\rho)))\Big)\,d\rho = 0.$$

Since $|s^+(\rho)| < 1,$ this is possible only if $\Phi(\rho)\equiv 0.$ Then $f(y)=0,$ and Theorem 2.8.6 is proved. □

Remark 2.8.1. The main equations (2.8.54)-(2.8.55) can be rewritten in the form

$$\left.\begin{aligned} &F^+(2x+y)+B^+(x,y)+\int_0^\infty B^+(x,t)F^+(2x+y+t)\,dt = 0,\ y>0,\\ &F^-(2x+y)+B^-(x,y)+\int_{-\infty}^0 B^-(x,t)F^-(2x+y+t)\,dt = 0,\ y<0, \end{aligned}\right\} \tag{2.8.74}$$

where $B^\pm(x,y) = A^\pm(x,x+y)$.

Using the main equations (2.8.54)-(2.8.55) we provide the solution of the inverse scattering problem of recovering the potential q from the given scattering data J^+ (or J^-). First we prove the uniqueness theorem.

Theorem 2.8.7. *The specification of the scattering data J^+ (or J^-) uniquely determines the potential q.*

Proof. Let J^+ and $\tilde J^+$ be the right scattering data for the potentials q and $\tilde q$ respectively, and let $J^+ = \tilde J^+$. Then, it follows from (2.8.56) and (2.8.28) that $F^+(x) = \tilde F^+(x)$. By virtue of Theorems 2.8.5 and 2.8.6, $A^+(x,y) = \tilde A^+(x,y)$. Therefore, taking (2.8.9) into account, we get $q = \tilde q$. For J^- the arguments are the same. □

The solution of the inverse scattering problem can be constructed by the following algorithm.

Algorithm 2.8.2. Let the scattering data J^+ (or J^-) be given. Then
1) Calculate the function $F^+(x)$ (or $F^-(x)$) by (2.8.56) and (2.8.28).
2) Find $A^+(x,y)$ (or $A^-(x,y)$) by solving the main equation (2.8.54) (or (2.8.55) respectively).
3) Construct $q(x) = -2\dfrac{d}{dx}A^+(x,x)$ (or $q(x) = 2\dfrac{d}{dx}A^-(x,x)$).

Let us now formulate necessary and sufficient conditions for the solvability of the inverse scattering problem.

Theorem 2.8.8. *For data* $J^+ = \{s^+(\rho),\ \lambda_k,\ \alpha_k^+;\ \rho \in \mathbf{R},\ k = \overline{1,N}\}$ *to be the right scattering data for a certain real potential* q *satisfying* $(2.8.2)$*, it is necessary and sufficient that the following conditions hold:*

1) $\lambda_k = -\tau_k^2,\ 0 < \tau_1 < \ldots < \tau_N;\ \alpha_k^+ > 0,\ k = \overline{1,N}.$

2) *For real* $\rho \neq 0$*, the function* $s^+(\rho)$ *is continuous,* $\overline{s^+(\rho)} = s^+(-\rho),\ |s^+(\rho)| < 1,$ *and*

$$s^+(\rho) = o\Big(\frac{1}{\rho}\Big) \ \text{as}\ |\rho| \to \infty,$$

$$\frac{\rho^2}{1 - |s^+(\rho)|^2} = O(1) \ \text{as}\ |\rho| \to 0.$$

3) *The function* $\rho(a(\rho) - 1)$*, where* $a(\rho)$ *is defined by*

$$a(\rho) := \prod_{k=1}^{N} \frac{\rho - i\tau_k}{\rho + i\tau_k} \exp(B(\rho)),$$

$$B(\rho) := -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln(1 - |s^+(\xi)|^2)}{\xi - \rho}\, d\xi,\ \rho \in \Omega_+,$$

is continuous and bounded in $\overline{\Omega_+}$*, and*

$$\frac{1}{a(\rho)} = O(1) \ \text{as}\ |\rho| \to 0,\ \rho \in \overline{\Omega_+},$$

$$\lim_{\rho \to 0} \rho a(\rho) \Big(s^+(\rho) + 1\Big) = 0 \ \text{for real}\ \rho.$$

4) *The functions* $R^{\pm}(x)$*, defined by*

$$R^{\pm}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} s^{\pm}(\rho) \exp(\pm i\rho x)\, d\rho, \quad s^-(\rho) := -s^+(-\rho) \frac{a(-\rho)}{a(\rho)},$$

are real and absolutely continuous, $R^{\pm}(x) \in L_2(-\infty, \infty)$*, and for each fixed* $a > -\infty$*,* $(2.8.72)$ *holds.*

The necessity part of Theorem 2.8.8 was proved above. For the sufficiency part see [27, Ch.3].

3. Reflectionless potentials. Modification of the discrete spectrum.

A potential q satisfying (2.8.2) is called *reflectionless* if $b(\rho) \equiv 0$. By virtue of (2.8.26) and (2.8.53) we have in this case

$$s^{\pm}(\rho) \equiv 0, \quad a(\rho) = \prod_{k=1}^{N} \frac{\rho - i\tau_k}{\rho + i\tau_k}. \tag{2.8.75}$$

Theorem 2.8.8 allows one to prove the existence of reflectionless potentials and to describe all of them. Namely, the following theorem is valid.

Theorem 2.8.9. *Let arbitrary numbers* $\lambda_k = -\tau_k^2 < 0,\ \alpha_k^+ > 0,\ k = \overline{1,N}$ *be given. Take* $s^+(\rho) \equiv 0,\ \rho \in \mathbf{R},$ *and consider the data* $J^+ = \{s^+(\rho),\ \lambda_k,\ \alpha_k^+;\ \rho \in \mathbf{R},\ k = \overline{1,N}\}$. *Then, there exists a unique reflectionless potential* q *satisfying* (2.8.2) *for which* J^+ *are the right scattering data.*

Theorem 2.8.9 is an obvious corollary of Theorem 2.8.8 since for this set J^+ all conditions of Theorem 2.8.8 are fulfilled and (2.8.75) holds.

For a reflectionless potential, the Gelfand-Levitan-Marchenko equation (2.8.54) takes the form

$$A^+(x,y) + \sum_{k=1}^{N} \alpha_k^+ \exp(-\tau_k(x+y)) + \sum_{k=1}^{N} \alpha_k^+ \exp(-\tau_k y) \int_x^\infty A^+(x,t) \exp(-\tau_k t)\, dt = 0. \tag{2.8.76}$$

We seek a solution of (2.8.76) in the form:

$$A^+(x,y) = \sum_{k=1}^{N} P_k(x) \exp(-\tau_k y).$$

Substituting this into (2.8.76), we obtain the following linear algebraic system with respect to $P_k(x)$:

$$P_k(x) + \sum_{j=1}^{N} \alpha_j^+ \frac{\exp(-(\tau_k+\tau_j)x)}{\tau_k+\tau_j} P_j(x) = -\alpha_k^+ \exp(-\tau_k x), \quad k = \overline{1,N}. \tag{2.8.77}$$

Solving (2.8.77) we infer $P_k(x) = \Delta_k(x)/\Delta(x)$, where

$$\Delta(x) = \det\left[\delta_{kl} + \alpha_k^+ \frac{\exp(-(\tau_k+\tau_l)x)}{\tau_k+\tau_l}\right]_{k,l=\overline{1,N}}, \tag{2.8.78}$$

and $\Delta_k(x)$ is the determinant obtained from $\Delta(x)$ by means of the replacement of k-column by the right-hand side column. Then

$$q(x) = -2\frac{d}{dx}A^+(x,x) = -2\sum_{k=1}^{N} \frac{\Delta_k(x)}{\Delta(x)} \exp(-\tau_k x),$$

and consequently one can show that

$$q(x) = -2\frac{d^2}{dx^2} \ln \Delta(x). \tag{2.8.79}$$

Thus, (2.8.78) and (2.8.79) allow one to calculate reflectionless potentials from the given numbers $\{\lambda_k,\ \alpha_k^+\}_{k=\overline{1,N}}$.

Example 2.8.2. Let $N = 1,\ \tau = \tau_1,\ \alpha = \alpha_1^+,$ and

$$\Delta(x) = 1 + \frac{\alpha}{2\tau} \exp(-2\tau x).$$

Then (2.8.79) gives

$$q(x) = -\frac{4\tau\alpha}{\left(\exp(\tau x) + \frac{\alpha}{2\tau}\exp(-\tau x)\right)^2}.$$

Denote

$$\beta = -\frac{1}{2\tau}\ln\frac{2\tau}{\alpha}.$$

Then

$$q(x) = -\frac{2\tau^2}{\cosh^2(\tau(x-\beta))}.$$

If $q = 0$, then $s^{\pm}(\rho) \equiv 0$, $N = 0$, $a(\rho) \equiv 1$. Therefore, Theorem 2.8.9 shows that all reflectionless potentials can be constructed from the zero potential and given data $\{\lambda_k, \alpha_k^+\}$, $k = \overline{1,N}$. Below we briefly consider a more general case of changing the discrete spectrum for an arbitrary potential q. Namely, the following theorem is valid.

Theorem 2.8.10. *Let $J^+ = \{s^+(\rho), \lambda_k, \alpha_k^+;\ \rho \in \mathbf{R}, k = \overline{1,N}\}$ be the right scattering data for a certain real potential q satifying (2.8.2). Take arbitrary numbers $\tilde\lambda_k = -\tilde\tau_k^2 < 0$, $\tilde\alpha_k^+ > 0$, $k = \overline{1,\tilde N}$, and consider the set $\tilde J^+ = \{s^+(\rho), \tilde\lambda_k, \tilde\alpha_k^+;\ \rho \in \mathbf{R}, k = \overline{1,\tilde N}\}$ with the same $s^+(\rho)$ as in J^+. Then there exists a real potential $\tilde q$ satisfying (2.8.2), for which $\tilde J^+$ represents the right scattering data.*

Proof. Let us check the conditions of Theorem 2.8.8 for $\tilde J^+$. For this purpose we construct the functions $\tilde a(\rho)$ and $\tilde s^-(\rho)$ by the formulae

$$\tilde a(\rho) := \prod_{k=1}^{\tilde N}\frac{\rho - i\tilde\tau_k}{\rho + i\tilde\tau_k}\exp\Big(-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\ln(1-|s^+(\xi)|^2)}{\xi-\rho}\,d\xi\Big),\ \rho \in \Omega_+,$$

$$\tilde s^-(\rho) := -s^+(-\rho)\frac{\tilde a(-\rho)}{\tilde a(\rho)}. \tag{2.8.80}$$

Together with (2.8.53) this yields

$$\tilde a(\rho) = a(\rho)\prod_{k=1}^{\tilde N}\frac{\rho - i\tilde\tau_k}{\rho + i\tilde\tau_k}\prod_{k=1}^{N}\frac{\rho + i\tau_k}{\rho - i\tau_k}. \tag{2.8.81}$$

By virtue of (2.8.26),

$$s^-(\rho) = -s^+(-\rho)\frac{a(-\rho)}{a(\rho)}. \tag{2.8.82}$$

Using (2.8.80)-(2.8.82) we get $\tilde s^-(\rho) = s^-(\rho)$. Since the scattering data J^+ satisfy all conditions of Theorem 2.8.8 it follows from (2.8.81) and (2.8.82) that $\tilde J^+$ also satisfy all conditions of Theorem 2.8.8. Then, by Theorem 2.8.8 there exists a real potential $\tilde q$ satisfying (2.8.2) for which $\tilde J^+$ are the right scattering data. □

2.9. The Cauchy Problem for the Korteweg-De Vries Equation

Inverse spectral problems play an important role for integrating some nonlinear evolution equations in mathematical physics. In 1967, G.Gardner, G.Green, M.Kruskal and R.Miura [28] found a deep connection of the well-known (from XIX century) nonlinear Korteweg-de Vries (KdV) equation

$$q_t = 6qq_x - q_{xxx}$$

with the spectral theory of Sturm-Liouville operators. They could manage to solve globally the Cauchy problem for the KdV equation by means of reduction to the inverse spectral problem. These investigations created a new branch in mathematical physics (for further discussions see [29]-[32]). In this section we provide the solution of the Cauchy problem for the KdV equation on the line. For this purpose we use ideas from [28], [30] and results of Section 2.8 on the inverse scattering problem for the Sturm-Liouville operator on the line.

Consider the Cauchy problem for the KdV equation on the line:

$$q_t = 6qq_x - q_{xxx}, \quad -\infty < x < \infty,\ t > 0, \tag{2.9.1}$$

$$q_{|t=0} = q_0(x), \tag{2.9.2}$$

where $q_0(x)$ is a real function such that $(1+|x|)|q_0(x)| \in L(0,\infty)$. Denote by Q_0 the set of real functions $q(x,t)$, $-\infty < x < \infty$, $t \ge 0$, such that for each fixed $T > 0$,

$$\max_{0\le t\le T} \int_{-\infty}^{\infty} (1+|x|)|q(x,t)|\, dt < \infty.$$

Let Q_1 be the set of functions $q(x,t)$ such that $q, \dot{q}, q', q'', q''' \in Q_0$. Here and below, "dot" denotes derivatives with respect to t, and "prime" denotes derivatives with respect to x. We will seek the solution of the Cauchy problem (2.9.1)-(2.9.2) in the class Q_1. First we prove the following uniqueness theorem.

Theorem 2.9.1. *The Cauchy problem* $(2.9.1)-(2.9.2)$ *has at most one solution.*

Proof. Let $q, \tilde{q} \in Q_1$ be solutions of the the Cauchy problem (2.9.1)-(2.9.2). Denote $w := q - \tilde{q}$. Then $w \in Q_1$, $w_{|t=0} = 0$, and

$$w_t = 6(qw_x + w\tilde{q}_x) - w_{xxx}.$$

Multiplying this equality by w and integrating with respect to x, we get

$$\int_{-\infty}^{\infty} ww_t\, dx = 6\int_{-\infty}^{\infty} w(qw_x + w\tilde{q}_x)\, dx - \int_{-\infty}^{\infty} ww_{xxx}\, dx.$$

Integration by parts yields

$$\int_{-\infty}^{\infty} ww_{xxx}\, dx = -\int_{-\infty}^{\infty} w_x w_{xx}\, dx = \int_{-\infty}^{\infty} w_x w_{xx}\, dx,$$

and consequently

$$\int_{-\infty}^{\infty} ww_{xxx}\, dx = 0.$$

Since

$$\int_{-\infty}^{\infty} qww_x\,dx = \int_{-\infty}^{\infty} q\Big(\frac{1}{2}w^2\Big)_x dx = -\frac{1}{2}\int_{-\infty}^{\infty} q_x w^2\,dx,$$

it follows that

$$\int_{-\infty}^{\infty} ww_t\,dx = \int_{-\infty}^{\infty} \Big(\tilde{q}_x - \frac{1}{2}q_x\Big)w^2\,dx.$$

Denote

$$E(t) = \frac{1}{2}\int_{-\infty}^{\infty} w^2\,dx, \quad m(t) = 12\max_{x\in\mathbf{R}}\Big|\tilde{q}_x - \frac{1}{2}q_x\Big|.$$

Then

$$\frac{d}{dt}E(t) \le m(t)E(t),$$

and consequently,

$$0 \le E(t) \le E(0)\exp\Big(\int_0^t m(\xi)\,d\xi\Big).$$

Since $E(0)=0$ we deduce $E(t)\equiv 0,$ i.e. $w\equiv 0.$ □

Our next goal is to construct the solution of the Cauchy problem (2.9.1)-(2.9.2) by reduction to the inverse scattering problem for the Sturm-Liouville equation on the line.

Let $q(x,t)$ be the solution of (2.9.1)-(2.9.2). Consider the Sturm-Liouville equation

$$Ly := -y'' + q(x,t)y = \lambda y \tag{2.9.3}$$

with t as a parameter. Then the Jost-type solutions of (2.9.3) and the scattering data depend on t. Let us show that equation (2.9.1) is equivalent to the equation

$$\dot{L} = [A,L], \tag{2.9.4}$$

where in this section

$$Ay = -4y''' + 6qy' + 3q'y$$

is a linear differential operator, and $[A,L] := AL - LA.$

Indeed, since $Ly = -y'' + qy,$ we have

$$\dot{L}y = \dot{q}y,\ ALy = -4(-y''+qy)''' + 6q(-y''+qy)' + 3q'(-y''+qy),$$

$$LAy = -(-4y''' + 6qy' + 3q'y)'' + q(-4y''' + 6qy' + 3q'y),$$

and consequently $(AL-LA)y = (6qq' - q''')y.$

Equation (2.9.4) is called the Lax equation or Lax representation, and the pair A,L is called the Lax pair, corresponding to (2.9.3).

Lemma 2.9.1. *Let $q(x,t)$ be a solution of (2.9.1), and let $y = y(x,t,\lambda)$ be a solution of (2.9.3). Then $(L-\lambda)(\dot{y} - Ay) = 0,$ i.e. the function $\dot{y} - Ay$ is also a solution of (2.9.3).*

Indeed, differentiating (2.9.3) with respect to $t,$ we get $\dot{L}y + (L-\lambda)\dot{y} = 0,$ or, in view of (2.9.4), $(L-\lambda)\dot{y} = LAy - ALy = (L-\lambda)Ay.$ □

Let $e(x,t,\rho)$ and $g(x,t,\rho)$ be the Jost-type solutions of (2.9.3) introduced in Section 2.8. Denote $e_{\pm} = e(x,t,\pm\rho),\ g_{\pm} = g(x,t,\pm\rho).$

Lemma 2.9.2. *The following relation holds*

$$\dot{e}_+ = Ae_+ - 4i\rho^3 e_+. \tag{2.9.5}$$

Proof. By Lemma 2.9.1, the function $\dot{e}_+ - Ae_+$ is a solution of (2.9.3). Since the functions $\{e_+, e_-\}$ form a fundamental system of solutions of (2.9.3), we have

$$\dot{e}_+ - Ae_+ = \beta_1 e_+ + \beta_2 e_-,$$

where $\beta_k = \beta_k(t,\rho)$, $k = 1,2$ do not depend on x. As $x \to +\infty$,

$$e_\pm \sim \exp(\pm i\rho x), \quad \dot{e}_+ \sim 0, \quad Ae_+ \sim 4i\rho^3 \exp(i\rho x),$$

consequently $\beta_1 = -4i\rho^3$, $\beta_2 = 0$, and (2.9.5) is proved. □

Lemma 2.9.3. *The following relations hold*

$$\dot{a} = 0, \quad \dot{b} = -8i\rho^3 b, \quad \dot{s}^+ = 8i\rho^3 s^+, \tag{2.9.6}$$

$$\dot{\lambda}_j = 0, \quad \dot{\alpha}_j^+ = 8\kappa_j^3 \alpha_j^+. \tag{2.9.7}$$

Proof. According to (2.8.18),

$$e_+ = ag_+ + bg_-. \tag{2.9.8}$$

Differentiating (2.9.8) with respect to t, we get $\dot{e}_+ = (\dot{a}g_+ + \dot{b}g_-) + (a\dot{g}_+ + b\dot{g}_-)$. Using (2.9.5) and (2.9.8) we calculate

$$a(Ag_+ - 4i\rho^3 g_+) + b(Ag_- - 4i\rho^3 g_-) = (\dot{a}g_+ + \dot{b}g_-) + (a\dot{g}_+ + b\dot{g}_-). \tag{2.9.9}$$

Since $g_\pm \sim \exp(\pm i\rho x)$, $\dot{g}_\pm \sim 0$, $Ag_\pm \sim \pm 4i\rho^3 \exp(\pm i\rho x)$ as $x \to -\infty$, then (2.9.9) yields $-8i\rho^3 \exp(-i\rho x) \sim \dot{a}\exp(i\rho x) + \dot{b}\exp(-i\rho x)$, i.e. $\dot{a} = 0$, $\dot{b} = -8i\rho^3 b$. Consequently, $\dot{s}^+ = 8i\rho^3 s^+$, and (2.9.6) is proved.

The eigenvalues $\lambda_j = \rho_j^2 = -\kappa_j^2$, $\kappa_j > 0$, $j = \overline{1,N}$ are the zeros of the function $a = a(\rho,t)$. Hence, according to $\dot{a} = 0$, we have $\dot{\lambda}_j = 0$. Denote

$$e_j = e(x,t,i\kappa_j), \; g_j = g(x,t,i\kappa_j), \quad j = \overline{1,N}.$$

By Theorem 2.8.3, $g_j = d_j e_j$, where $d_j = d_j(t)$ do not depend on x. Differentiating the relation $g_j = d_j e_j$ with respect to t and using (2.9.5), we infer

$$\dot{g}_j = \dot{d}_j e_j + d_j \dot{e}_j = \dot{d}_j e_j + d_j Ae_j - 4\kappa_j^3 d_j e_j$$

or

$$\dot{g}_j = \frac{\dot{d}_j}{d_j} g_j + Ag_j - 4\kappa_j^3 g_j.$$

As $x \to -\infty$, $g_j \sim \exp(\kappa_j x)$, $\dot{g}_j \sim 0$, $Ag_j \sim -4\kappa_j^3 \exp(\kappa_j x)$, and consequently $\dot{d}_j = 8\kappa_j^3 d_j$, or, in view of (2.8.32), $\dot{\alpha}_j^+ = 8\kappa_j^3 \alpha_j^+$. □

Thus, it follows from Lemma 2.9.3 that we have proved the following theorem.

Theorem 2.9.2. *Let* $q(x,t)$ *be the solution of the Cauchy problem* $(2.9.1)-(2.9.2)$, *and let* $J^+(t)=\{s^+(t,\rho),\ \lambda_j(t),\ \alpha_j^+(t),\ j=\overline{1,N}\}$ *be the scattering data for* $q(x,t)$. *Then*

$$\left.\begin{aligned} s^+(t,\rho)&=s^+(0,\rho)\exp(8i\rho^3 t),\\ \lambda_j(t)=\lambda_j(0),\quad \alpha_j^+(t)&=\alpha_j^+(0)\exp(8\kappa_j^3 t),\ j=\overline{1,N}\quad (\lambda_j=-\kappa_j^2). \end{aligned}\right\} \tag{2.9.10}$$

The formulae (2.9.10) give us the evolution of the scattering data with respect to t, and we obtain the following algorithm for the solution of the Cauchy problem (2.9.1)-(2.9.2).

Algorithm 2.9.1. *Let the function* $q(x,0)=q_0(x)$ *be given. Then*
1) *construct the scattering data* $\{s^+(0,\rho),\ \lambda_j(0),\ \alpha_j^+(0),\ j=\overline{1,N}\}$;
2) *calculate* $\{s^+(t,\rho),\ \lambda_j(t),\ \alpha_j^+(t),\ j=\overline{1,N}\}$ *by* (2.9.10);
3) *find the function* $q(x,t)$ *by solving the inverse scattering problem* (*see Section* 2.8).

We notice once more the main points for the solution of the Cauchy problem (2.9.1)-(2.9.2) by the inverse problem method:
(1) The presence of the Lax representation (2.9.4).
(2) The evolution of the scattering data with respect to t.
(3) The solution of the inverse problem.

Among the solutions of the KdV equation (2.9.1) there are very important particular solutions of the form $q(x,t)=f(x-ct)$. Such solutions are called *solitons*. Substituting $q(x,t)=f(x-ct)$ into (2.9.1), we get $f'''+6ff'+cf'=0$, or $(f''+3f^2+cf)'=0$. Clearly, the function

$$f(x)=-\frac{c}{2\cosh^2\left(\frac{\sqrt{c}x}{2}\right)}$$

satisfies this equation. Hence, the function

$$q(x,t)=-\frac{c}{2\cosh^2\left(\frac{\sqrt{c}(x-ct)}{2}\right)} \tag{2.9.11}$$

is a soliton (see fig. 2.9.1).

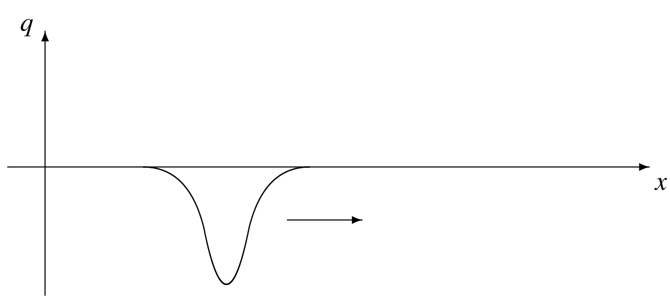

Figure 2.9.1.

It is interesting to note that solitons correspond to reflectionless potentials (see Section 2.8). Consider the Cauchy problem (2.9.1)-(2.9.2) in the case when $q_0(x)$ is a reflectionless potential, i.e. $s^+(0,\rho)=0.$ Then, by virtue of (2.9.10), $s^+(t,\rho)=0$ for all $t,$ i.e. the solution $q(x,t)$ of the Cauchy problem (2.9.1)-(2.9.2) is a reflectionless potential for all t. Using (2.9.10) and (2.8.79), we derive

$$q(x,t)=-2\frac{d^2}{dx^2}\Delta(x,t),$$

$$\Delta(x,t)=\det\left[\delta_{kl}+\alpha_k^+(0)\exp(8\kappa_k^3 t)\frac{\exp(-(\kappa_{k+}\kappa_l)x)}{\kappa_{k+}\kappa_l}\right]_{k,l=\overline{1,N}}.$$

In particular, if $N=1,$ then $\alpha_1^+(0)=2\kappa_1,$ and

$$q(x,t)=-\frac{2\kappa_1^2}{\cosh^2(\kappa_1(x-4\kappa_1^2 t))}.$$

Taking $c=4\kappa_1^2,$ we get (2.9.11).

Chapter 3.

Parabolic Partial Differential Equations

Parabolic partial differential equations usually describe various diffusion processes. The most important equation of parabolic type is the *heat equation* or *diffusion equation* (see Section 1.1). The properties of the solutions of parabolic equations do not depend essentially on the dimension of the space, and therefore we confine ourselves to considerations concerning the case of one spatial variable. In Section 3.1 we study the mixed problem for the heat equation on a finite interval. Section 3.2 deals with the Cauchy problem on the line for the heat equation. We note that in Chapter 6, Section 6.3 one can find exercises and illustrative example for problems related to parabolic partial differential equations.

3.1. The Mixed Problem for the Heat Equation

We consider the following mixed problem

$$u_t = u_{xx}, \quad 0 < x < l,\ t > 0, \tag{3.1.1}$$

$$u_{|x=0} = u_{|x=l} = 0, \tag{3.1.2}$$

$$u_{|t=0} = \varphi(x). \tag{3.1.3}$$

This problem describes the one-dimensional propagation of heat in a finite slender homogeneous rod which lies along the x-axis, provided that the ends of the rod $x = 0$ and $x = l$ have zero temperature, and the initial temperature $\varphi(x)$ is known.
Denote by $D = \{(x,t) :\ 0 < x < l, t > 0\}$, $\Gamma = \partial D$ the boundary of the domain D.

Definition 3.1.1. A function $u(x,t)$ is called a solution of problem (3.1.1)-(3.1.3) if $u(x,t) \in C(\overline{D})$, $u(x,t) \in C^2(D)$, and $u(x,t)$ satisfies (3.1.1)-(3.1.3).

First we will prove the maximum (and minimum) principle for the heat conduction equation. Denote $D_T = \{(x,t) :\ 0 < x < l, 0 < t \le T\}$, $\Gamma_T = \Gamma \cap \overline{D_T}$ (i.e. Γ_T is the part of the boundary of D_T without the upper cap (see fig. 3.1.1).

Figure 3.1.1.

Theorem 3.1.1. *Let* $u(x,t) \in C(\overline{D_T})$, $u(x,t) \in C^2(D_T)$, *and let* $u(x,t)$ *satisfy equation* $(3.1.1)$ *in* D_T. *Then* $u(x,t)$ *attains its maximum and minimum on* Γ_T.

Proof. For definiteness, we give the proof for the maximum. Denote

$$M = \max_{\overline{D_T}} u(x,t), \quad m = \max_{\Gamma_T} u(x,t).$$

Clearly, $M \geq m$. We have to prove that $M = m$. Suppose that $M > m$. Let $M = u(x_0,t_0)$, i.e. the maximum is attained at the point (x_0,t_0). Consider the function

$$v(x,t) = u(x,t) + \frac{M-m}{4l^2} x^2.$$

Then

$$v(x_0,t_0) \geq M,$$

$$v_{|\Gamma_T} \leq m + \frac{M-m}{4l^2} l^2 = \frac{M}{4} + \frac{3m}{4} < M,$$

and consequently, the maximum of the functions $v(x,t)$ is not attained on Γ_T. Let

$$\max_{\overline{D_T}} v(x,t) = v(x_1,t_1),$$

and $(x_1,t_1) \notin \Gamma_T$. According to the necessary conditions for an maximum we have

$$v_t(x_1,t_1) \geq 0,\ v_x(x_1,t_1) = 0,\ v_{xx}(x_1,t_1) \leq 0.$$

On the other hand,

$$v_t - v_{xx} = (u_t - u_{xx}) - \frac{M-m}{2l^2} < 0.$$

This contradiction proves the theorem. □

As an immediate consequence of the maximum (and minimum) principle we get the following interesting corollaries.

Corollary 3.1.1 (the comparison principle). *Let* $u_1(x,t), u_2(x,t) \in C(\overline{D}) \cap C^2(D)$ *satisfy* $(3.1.1)$. *If* $u_1(x,t) \leq u_2(x,t)$ *on* Γ, *then* $u_1(x,t) \leq u_2(x,t)$ *in* $\overline{D}$.

Corollary 3.1.2 (the uniqueness theorem). *If the solution of the problem* $(3.1.1)-(3.1.3)$ *exists, then it is unique.*

Indeed, let $u_1(x,t)$ and $u_2(x,t)$ be solutions of (3.1.1)-(3.1.3). Then $u_1(x,t)=u_2(x,t)$ on Γ, and consequently, $u_1(x,t)=u_2(x,t)$ in $\overline{D}$.

Corollary 3.1.3 (the stability theorem). *Let* $u(x,t)$ *and* $\tilde{u}(x,t)$ *be solutions of* $(3.1.1)-(3.1.3)$ *under the initial conditions* φ *and* $\tilde{\varphi}$, *respectively. If* $|\varphi(x)-\tilde{\varphi}(x)|\le\varepsilon$, $0\le x\le l$, *then* $|u(x,t)-\tilde{u}(x,t)|\le\varepsilon$ *in* $\overline{D}$.

Indeed, denote $v(x,t)=u(x,t)-\tilde{u}(x,t)$. Then $v(x,t)\in C(\bar{D})$, satisfies equation (3.1.1), and $|v(x,t)|\le\varepsilon$ on Γ. Hence $|v(x,t)|\le\varepsilon$ in $\overline{D}$.

We will solve the mixed problem (3.1.1)-(3.1.3) by the method of separation of variables. First we consider the following auxiliary problem. We will seek non-trivial (i.e. not identically equal to zero) solutions of equation (3.1.1) satisfying the boundary condition (3.1.2) and having the form

$$u(x,t)=Y(x)T(t).$$

Acting in the same way as in Section 2.2, we obtain

$$\frac{Y''(x)}{Y(x)}=\frac{\dot{T}(t)}{T(t)}=-\lambda,$$

where λ is a complex parameter. Hence

$$\dot{T}(t)+\lambda T(t)=0, \tag{3.1.4}$$

$$Y''(x)+\lambda Y(x)=0, \quad Y(0)=Y(l)=0. \tag{3.1.5}$$

In Section 2.2 we found the eigenvalues

$$\lambda_n=\left(\frac{\pi n}{l}\right)^2, \quad n\ge 1,$$

and the eigenfunctions

$$Y_n(x)=\sin\frac{\pi n}{l}x, \quad n\ge 1,$$

of the Sturm-Liouville problem (3.1.5). Equation (3.1.4) for $\lambda=\lambda_n$ has the general solution

$$T_n(t)=A_n\exp(-\lambda_n t),$$

where A_n are arbitrary constants. Thus, the solutions of the auxiliary problem have the form

$$u_n(x,t)=A_n\exp\left(-\left(\frac{\pi n}{l}\right)^2 t\right)\sin\frac{\pi n}{l}x, \quad n\ge 1. \tag{3.1.6}$$

We will seek the solution of the mixed problem (3.1.1)-(3.1.3) by superposition of functions of the form (3.1.6):

$$u(x,t)=\sum_{n=1}^{\infty}A_n\exp\left(-\left(\frac{\pi n}{l}\right)^2 t\right)\sin\frac{\pi n}{l}x. \tag{3.1.7}$$

Formally, by construction, the function $u(x,t)$ satisfies equation (3.1.1) and the boundary conditions (3.1.2) for any A_n. Choose A_n such that $u(x,t)$ satisfies the initial condition (3.1.3). Substituting (3.1.7) into (3.1.3) we get

$$\varphi(x) = \sum_{n=1}^{\infty} A_n \sin\frac{\pi n}{l}x.$$

Using the formulas for the Fourier coefficients we calculate (formally)

$$A_n = \frac{2}{l}\int_0^l \varphi(x)\sin\frac{\pi n}{l}x\,dx, \quad n \ge 1. \tag{3.1.8}$$

Theorem 3.1.2. *Let* $\varphi(x) \in C[0,l]$, $\varphi(0) = \varphi(l) = 0$. *Then the solution of the mixed problem* $(3.1.1) - (3.1.3)$ *exists, is unique and is given by the formulae* $(3.1.7) - (3.1.8)$.

Proof. It is sufficient to prove that the function $u(x,t)$, defined by (3.1.7)-(3.1.8), is a solution of problem (3.1.1)-(3.1.3). Fix $\delta > 0$ and consider the domains $\Omega_\delta = \{(x,t) : 0 \le x \le l, t \ge \delta\}$ and $\Omega = \{(x,t) : 0 \le x \le l, t > 0\}$. Since $|A_n| \le C$, we have

$$\sum_{n=1}^{\infty} |A_n| n^s \exp\left(-\left(\frac{\pi n}{l}\right)^2 \delta\right) < \infty \quad \text{for all} \quad s \ge 0. \tag{3.1.9}$$

It follows from (3.1.9) that the function $u(x,t)$, defined by (3.1.7)-(3.1.8), is infinitely differentiable in Ω_δ, i.e. it has in Ω_δ continuous partial derivatives of all orders, and these derivatives can be obtained by termwise differentiation of the series (3.1.7). By virtue of the arbitrariness of δ we deduce that $u(x,t) \in C^\infty(\Omega)$. Clearly, $u(x,t)$ satisfies (3.1.1) and (3.1.2). It is more complicated to deal with the initial condition (1.3.3), since in the general case the series (3.1.7) for $t = 0$ can be divergent.

1) First we consider the particular case when $\varphi(x) \in C^1[0,l]$, $\varphi(0) = \varphi(l) = 0$. Then, using in (3.1.8) integration by parts (as in Section 2.2), we obtain

$$\sum_{n=1}^{\infty} |A_n| < \infty.$$

Consequently, the series (3.1.7) converges absolutely and uniformly in $\overline{D}$, and $u(x,t) \in C(\overline{D})$. For $t = 0$ we have

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{\pi n}{l}x,$$

hence

$$A_n = \frac{2}{l}\int_0^l u(x,0)\sin\frac{\pi n}{l}x\,dx.$$

Comparing this relation with (3.1.8) we get

$$\int_0^l (u(x,0) - \varphi(x))\sin\frac{\pi n}{l}x\,dx = 0, \quad n \ge 1.$$

On account of the completeness of the system $\{\sin\frac{\pi n}{l}x\}_{n\ge 1}$, we deduce that $u(x,0) = \varphi(x)$. Thus, for the case $\varphi(x) \in C^1[0,l]$ Theorem 3.1.2 is proved.

2) Consider now the general case when $\varphi(x) \in C[0,l]$, $\varphi(0) = \varphi(l) = 0$. Construct a sequence of the functions $\varphi_m(x) \in C^1[0,l]$, $\varphi_m(0) = \varphi_m(l) = 0$ such that

$$\lim_{m\to\infty} \max_{0\le x\le l} |\varphi_m(x) - \varphi(x)| = 0.$$

Let $u_m(x,t)$ be the solution of problem (3.1.1)-(3.1.3) with the initial data $\varphi_m(x)$. Using Corollary 3.1.3 we obtain that in $\overline{D}$ the sequence $u_m(x,t)$ converges uniformly to some function $\tilde{u}(x,t)$:

$$\lim_{m\to\infty} \max_{\overline{D}} |u_m(x,t) - \tilde{u}(x,t)| = 0,$$

and $\tilde{u}(x,t) \in C(\overline{D})$. Moreover,

$$\tilde{u}(x,0) = \varphi(x),$$

since

$$\varphi(x) = \lim_{m\to\infty} \varphi_m(x) = \lim_{m\to\infty} u_m(x,0) = \tilde{u}(x,0).$$

In the domain Ω_δ we rewrite (3.1.7) for $u(x,t)$ as follows:

$$u(x,t) = \sum_{n=1}^{\infty} \left(\frac{2}{l} \int_0^l \varphi(\xi) \sin\frac{\pi n}{l} \xi \, d\xi \right) \exp\left(-\left(\frac{\pi n}{l}\right)^2 t \right) \sin\frac{\pi n}{l} x,$$

and consequently,

$$u(x,t) = \int_0^l G(x,\xi,t)\varphi(\xi)\, d\xi,$$

where

$$G(x,\xi,t) = \frac{2}{l} \sum_{n=1}^{\infty} \exp\left(-\left(\frac{\pi n}{l}\right)^2 t \right) \sin\frac{\pi n}{l} \xi \sin\frac{\pi n}{l} x.$$

The function $G(x,\xi,t)$ is called the *Green's function*. Furthermore, in Ω_δ we have

$$\tilde{u}(x,t) = \lim_{m\to\infty} u_m(x,t) = \lim_{m\to\infty} \int_0^l G(x,\xi,t)\varphi_m(\xi)\, d\xi$$

$$= \int_0^l G(x,\xi,t) \lim_{m\to\infty} \varphi_m(\xi)\, d\xi = \int_0^l G(x,\xi,t)\varphi(\xi)\, d\xi = u(x,t).$$

By virtue of the arbitrariness of $\delta > 0$, we get $\tilde{u}(x,t) \equiv u(x,t)$ in Ω, hence (after defining $u(x,t)$ continuously for $t = 0$) $u(x,t) \in C(\overline{D})$, $u(x,0) = \varphi(x)$. Theorem 3.1.2 is proved. □

Definition 3.1.2. The solution of the mixed problem (3.1.1)-(3.1.3) is called stable if for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $|\varphi(x) - \tilde{\varphi}(x)| \le \delta$, $0 \le x \le l$, then $|u(x,t) - \tilde{u}(x,t)| \le \varepsilon$ in $\overline{D}$. Here $\tilde{u}(x,t)$ is the solution of the mixed problem with the initial condition $\tilde{u}(x,0) = \tilde{\varphi}(x)$.

It follows from Corollary 3.1.3 that the solution of the mixed problem (3.1.1)-(3.1.3) is stable (one can take $\delta = \varepsilon$). Thus, the mixed problem (3.1.1)-(3.1.3) is well-posed.

3.2. The Cauchy Problem for the Heat Equation

We consider the following Cauchy problem

$$u_t = u_{xx}, \quad -\infty < x < \infty,\ t > 0, \tag{3.2.1}$$

$$u_{|t=0} = \varphi(x). \tag{3.2.2}$$

This problem describes the one-dimensional propagation of heat in a infinite slender homogeneous rod which lies along the x-axis, provided that the initial temperature $\varphi(x)$ is known.

Denote $Q = \{(x,t) : \ -\infty < x < \infty,\ t > 0\}$.

Definition 3.2.1. A function $u(x,t)$ is called a solution of the Cauchy problem (3.2.1)-(3.2.2), if $u(x,t)$ is continuous and bounded in $\overline{Q}$, $u(x,t) \in C^2(Q)$, and $u(x,t)$ satisfies (3.2.1)-(3.2.2).

Theorem 3.2.1. *If a solution of problem* $(3.2.1)-(3.2.2)$ *exists, then it is unique.*

Proof. Let $u_1(x,t)$ and $u_2(x,t)$ be solutions of (3.2.1)-(3.2.2). Denote $v(x,t) = u_1(x,t) - u_2(x,t)$. Then $v(x,t)$ is continuous and bounded in $\overline{Q}$, $v_t = v_{xx}$ in Q, and $v(x,0) = 0$. Denote $M = \sup\limits_{\overline{Q}} v(x,t)$. Fix $L > 0$. Let $Q^0 = \{(x,t) : \ |x| < L, t > 0\}$, $\Gamma^0 = \partial Q^0$ is the boundary of Q^0. Consider the function

$$w(x,t) = \frac{M}{L^2}(x^2 + 2t).$$

We have

$$\left.\begin{aligned} |v(x,0)| = 0 \le w(x,0), \\ |v(\pm L,t)| \le M \le w(\pm L,t). \end{aligned}\right\} \tag{3.2.3}$$

The function $w(x,t)$ satisfies equation (3.2.1), and by virtue of (3.2.3),

$$-w(x,t) \le v(x,t) \le w(x,t)$$

on Γ^0. According to Corollary 3.1.1, the last inequality is also valid in $\overline{Q^0}$. Thus,

$$-\frac{M}{L^2}(x^2+2t) \le v(x,t) \le \frac{M}{L^2}(x^2+2t), \quad |x| \le L,\ t \ge 0.$$

As $L \to \infty$ we obtain $v(x,t) \equiv 0$, and Theorem 3.2.1 is proved. □

Derivation of the Poisson formula. We will seek bounded particular solutions of equation (3.2.1) which have the form $u(x,t) = Y(x)T(t)$. Then

$$Y(x)\dot{T}(t) = Y''(x)T(t)$$

or

$$\frac{Y''(x)}{Y(x)} = \frac{\dot{T}(t)}{T(t)} = -\lambda^2.$$

Thus, for the functions $Y(x)$ and $T(t)$ we get the ordinary differential equations:

$$\dot{T}(t)+\lambda^2 T(t)=0,$$

$$Y''(x)+\lambda^2 Y(x)=0.$$

The general solutions of these equations have the form

$$Y(x)=A_1(\lambda)e^{i\lambda x}+A_2(\lambda)e^{-i\lambda x},$$

$$T(t)=A_3(\lambda)e^{-\lambda^2 t}.$$

Therefore, the functions

$$u(x,t,\lambda)=A(\lambda)e^{-\lambda^2 t+i\lambda x}$$

are the desired bounded particular solutions of (3.2.1) admitting separation of variables.

We will seek the solution of the Cauchy problem (3.2.1)-(3.2.2) in the form

$$u(x,t)=\int_{-\infty}^{\infty}A(\lambda)e^{-\lambda^2 t+i\lambda x}\,d\lambda.$$

We determine $A(\lambda)$ from the initial condition (3.2.2). For $t=0$ we have

$$\varphi(x)=\int_{-\infty}^{\infty}A(\lambda)e^{i\lambda x}\,d\lambda.$$

Using the formulae for the Fourier transform (see [11]) we obtain (formally)

$$A(\lambda)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi(\xi)e^{-i\lambda\xi}\,d\xi,$$

and consequently,

$$u(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\varphi(\xi)e^{-i\lambda\xi}\,d\xi\right)e^{-\lambda^2 t+i\lambda x}\,d\lambda$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi(\xi)\left(\int_{-\infty}^{\infty}e^{-\lambda^2 t+i\lambda(x-\xi)}\,d\lambda\right)d\xi.$$

Let us calculate the inner integral. For this purpose we consider for a fixed t the function

$$g(y)=\int_{-\infty}^{\infty}e^{-\lambda^2 t+\lambda y}\,d\lambda. \tag{3.2.4}$$

Differentiating (3.2.4) and using integration by parts we deduce

$$g'(y)=\int_{-\infty}^{\infty}\lambda e^{-\lambda^2 t}e^{\lambda y}\,d\lambda=\frac{1}{2}\int_{-\infty}^{\infty}e^{\lambda y}e^{-\lambda^2 t}\,d(\lambda^2)$$

$$=-\frac{1}{2t}\Big|_{-\infty}^{\infty}e^{\lambda y}e^{-\lambda^2 t}+\frac{1}{2t}\int_{-\infty}^{\infty}ye^{\lambda y}e^{-\lambda^2 t}\,d\lambda=\frac{y}{2t}g(y).$$

Therefore,

$$g'(y)=\frac{y}{2t}g(y),$$

hence

$$g(y) = g(0)e^{\frac{y^2}{4t}}. \tag{3.2.5}$$

By virtue of (3.2.4),

$$g(0) = \int_{-\infty}^{\infty} e^{-\lambda^2 t}\,d\lambda = \frac{1}{\sqrt{t}}\int_{-\infty}^{\infty} e^{-\xi^2}\,d\xi = \sqrt{\frac{\pi}{t}}.$$

Thus, we conclude that

$$u(x,t) = \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} \varphi(\xi)e^{-\frac{(x-\xi)^2}{4t}}\,d\xi. \tag{3.2.6}$$

Formula (3.2.6) is called the *Poisson formula*, and the integral in (3.2.6) is called the *Poisson integral*.

Theorem 3.2.2. *Let the function $\varphi(x)$ be continuous and bounded. Then the solution of the Cauchy problem $(3.2.1)-(3.2.2)$ exists, is unique and is given by $(3.2.6)$.*

Proof. It is sufficient to prove that the function $u(x,t)$, defined by (3.2.6), is a solution of problem (3.2.1)-(3.2.2).

1) Since $|\varphi(\xi)| \le M$, we have

$$|u(x,t)| \le \frac{M}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4t}}\,d\xi = \frac{M}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\alpha^2}\,d\alpha = M,$$

where

$$\alpha = \frac{\xi - x}{2\sqrt{t}}.$$

Therefore, the Poisson integral converges absolutely and uniformly in $\overline{Q}$, and the function $u(x,t)$ is continuous and bounded in $\overline{Q}$.

2) The replacement $\alpha = \dfrac{\xi - x}{2\sqrt{t}}$ in (3.2.6) yields

$$u(x,t) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \varphi(x + 2\alpha\sqrt{t})e^{-\alpha^2}\,d\alpha. \tag{3.2.7}$$

For $t = 0$ we get

$$u(x,0) = \varphi(x)\cdot\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\alpha^2}\,d\alpha = \varphi(x),$$

i.e. the function $u(x,t)$ satisfies the initial condition (3.2.2).

3) Fix $\delta > 0$ and consider the domain $G_\delta = \{(x,t):\ -\infty < x < \infty, t \ge \delta\}$. In G_δ the function $u(x,t)$, defined by (3.2.6), is infinitely differentiable, i.e. it has in G_δ continuous partial derivatives of all orders, and these derivatives can be obtained by differentiation under the sign of integration. In particular,

$$u_t(x,t) = \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty} \varphi(\xi)e^{-\frac{(x-\xi)^2}{4t}}\left(-\frac{1}{2t\sqrt{t}} + \frac{(x-\xi)^2}{4t^2\sqrt{t}}\right)d\xi,$$

$$u_x(x,t) = \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} \varphi(\xi)e^{-\frac{(x-\xi)^2}{4t}}\left(-\frac{x-\xi}{2t}\right)d\xi,$$

$$u_{xx}(x,t) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \varphi(\xi) e^{-\frac{(x-\xi)^2}{4t}} \left(-\frac{1}{2t\sqrt{t}} + \frac{(x-\xi)^2}{4t^2\sqrt{t}} \right) d\xi.$$

Consequently, the function $u(x,t)$ satisfies equation (3.2.1) in G_δ.

By virtue of the arbitrariness of $\delta > 0$ we conclude that $u(x,t)$ satisfies (3.2.1) in Q. Theorem 3.2.2 is proved. □

Definition 3.2.2. The solution of the Cauchy problem (3.2.1)-(3.2.2) is called stable if for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $|\varphi(x) - \tilde{\varphi}(x)| \leq \delta$ for all x, then $|u(x,t) - \tilde{u}(x,t)| \leq \varepsilon$ in $\overline{Q}$.

It follows from (3.2.7) that

$$|u(x,t) - \tilde{u}(x,t)| \leq \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} |\varphi(x+2\alpha\sqrt{t}) - \tilde{\varphi}(x+2\alpha\sqrt{t})| e^{-\alpha^2} \, d\alpha$$

$$\leq \sup_x |\varphi(x) - \tilde{\varphi}(x)|,$$

and consequently, the solution of the Cauchy problem (3.2.1)-(3.2.2) is stable (one can take $\delta = \varepsilon$). Thus, the Cauchy problem (3.2.1)-(3.2.2) is well-posed.

Chapter 4.

Elliptic Partial Differential Equations

Elliptic equations usually describe stationary fields, for example, gravitational, electrostatical and temperature fields. The most important equations of elliptic type are the Laplace equation $\Delta u = 0$ and the Poisson equation $\Delta u = f(x)$, where $x = (x_1, \dots, x_n)$ are spatial variables, $u(x)$ is an unknown function,

$$\Delta u := \sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2}$$

is the Laplace operator (or Laplacian), and $f(x)$ is a given function. In this chapter we study boundary value problems for elliptic partial differential equations and present methods for their solutions.

4.1. Harmonic Functions and Their Properties

1. Basic notions

Let $D \subset \mathbf{R}^n$ be a bounded domain with the boundary Σ.

Definition 4.1.1. A surface Σ is called piecewise-smooth ($\Sigma \in PC^1$), if it consists of a finite number of pieces with continuous tangent planes on each of them, and if each straight line intersects Σ at no more than a finite number of points and/or segments.

Examples of surfaces of the class PC^1 are a ball, a parallelepiped, an ellipsoid, etc.. Everywhere below we assume that $\Sigma \in PC^1$. As before, in the sequel the notation $u(x) \in C^m(D)$ means that the function $u(x)$ has in D continuous partial derivatives up to the order m.

Definition 4.1.2. Let $u \in C^1(D), x \in \Sigma$, n_x be the outer normal to Σ at the point x. If uniformly on Σ there exists a finite limit

$$\lim_{\substack{y \to x \\ y = x - \alpha n_x, \alpha > 0}} \frac{\partial u(y)}{\partial n_x} := \frac{\partial u(x)}{\partial n_x},$$

we shall say that the function $u(x)$ has on Σ the *normal derivative* $\dfrac{\partial u(x)}{\partial n_x}$ (notation: $u(x) \in C^1_-(\overline{D})$).

Remark 4.1.1. 1) Clearly, if $u(x) \in C^1(\overline{D})$, then $u(x) \in C^1_-(\overline{D})$.
2) If $u(x) \in C^1_-(\overline{D})$, then $u(x) \in C(\overline{D})$ (after defining u on ∂D by continuity if necessary).

Definition 4.1.3. A function $u(x)$ is called *harmonic* in the domain D, if $u(x) \in C^2(D)$ and $\Delta u = 0$ in D.

Examples. 1) Let $n = 1$. Then $\Delta u = u''(x)$, and consequently, the harmonic functions are the linear ones: $u(x) = ax + b$.

2) Let $n = 2$, i.e. $x = (x_1, x_2)$, and let $z = x_1 + ix_2$. If the function $f(z) = u(x_1, x_2) + iv(x_1, x_2)$ is analytic in D, then by virtue of the Cauchy-Riemann conditions

$$\frac{\partial u}{\partial x_1} = \frac{\partial v}{\partial x_2}, \quad \frac{\partial u}{\partial x_2} = -\frac{\partial v}{\partial x_1},$$

the functions u and v are harmonic in D (and they are called conjugate harmonic functions). Conversely, if $u(x_1, x_2)$ is harmonic in D, then there exists a conjugate harmonic function $v(x_1, x_2)$ such that the function $f = u + iv$ is analytic in D, and

$$v(x_1, x_2) = \int_{(x_1^0, x_2^0)}^{(x_1, x_2)} \left(-\frac{\partial u}{\partial x_2} dx_1 + \frac{\partial u}{\partial x_1} dx_2 \right) + C$$

(the integral does not depend on the way of the integration since under the integral we have the total differential of v).

Fix $x^0 = (x_1^0, x_2^0)$ and denote

$$r = \|x - x^0\| = \sqrt{(x_1 - x_1^0)^2 + (x_2 - x_2^0)^2}.$$

It is easy to check that the function $u(x) = \ln \dfrac{1}{r}$ is harmonic everywhere except at the point x^0. This function is called the *fundamental solution* of the Laplace equation on the plane.

3) Let $n = 3$, i.e. $x = (x_1, x_2, x_3)$. Fix $x^0 = (x_1^0, x_2^0, x_3^0)$ and denote

$$r = \|x - x^0\| = \sqrt{\sum_{k=1}^{3} (x_k - x_k^0)^2}.$$

Then the function $u(x) = \dfrac{1}{r}$ is harmonic everywhere except at the point x^0 (see Lemma 2.5.1). This function is called the fundamental solution of the Laplace equation in the space $\mathbf{R}^3$.

Remark 4.1.2. The importance of the fundamental solution connects with the isotropy of the space when the physical picture depends only on the distance from the source of energy but not on the direction. For example, the function $u(x) = \dfrac{1}{r}$ represents the potential of the gravitational (electro-statical) field created by a point unit mass (point unit charge). Similar sense has the fundamental solution of the Laplace equation on the plane: this is the potential of the field produced by a charge of constant linear density $q = 1$, distributed uniformly along the line $x_1 = x_1^0,\, x_2 = x_2^0$.

2. Properties of harmonic functions

Let for definiteness $n = 3$.

Theorem 4.1.1. *Let* $u(x), v(x) \in C^2(D) \cap C^1_-(\overline{D})$. *Then*

$$\int_D (u\Delta v - v\Delta u)\, dx = \int_\Sigma \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) ds, \tag{4.1.1}$$

where n *is the outer normal to the surface* Σ. *Formula* (4.1.1) *is called Green's formula.*

Proof. Let D_α be the domain that is obtained from D by removing the balls $K_\alpha(x)$, $x \in \Sigma$ of radius α around the point x. Let Σ_α be the boundary of D_α. Applying the Gauß-Ostrogradskii formula for the domain D_α, we get

$$\int_{D_\alpha} (u\Delta v - v\Delta u)\, dx = \int_{D_\alpha} \left(\sum_{k=1}^{3} \frac{\partial}{\partial x_k} (uv_{x_k} - vu_{x_k}) \right) dx$$

$$= \int_{\Sigma_\alpha} \left(\sum_{k=1}^{3} (uv_{x_k} - vu_{x_k}) \cos(n, x_k) \right) ds$$

$$= \int_{\Sigma_\alpha} \left(u \sum_{k=1}^{3} v_{x_k} \cos(n, x_k) - v \sum_{k=1}^{3} u_{x_k} \cos(n, x_k) \right) ds$$

$$= \int_{\Sigma_\alpha} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) ds.$$

As $\alpha \to \infty$ this yields (4.1.1). □

Theorem 4.1.2. *Let* $u(x)$ *be harmonic in* D *and* $u(x) \in C^1_-(\overline{D})$. *Then*

$$\int_\Sigma \frac{\partial u}{\partial n}\, ds = 0. \tag{4.1.2}$$

Indeed, (4.1.2) follows from (4.1.1) for $v \equiv 1$.

Theorem 4.1.3. *Let* $u(x) \in C^2(D)$ *and assume that for any closed surface* $S \in PC^1$ *from* D *one has*

$$\int_S \frac{\partial u}{\partial n}\, ds = 0.$$

Then $u(x)$ *is a harmonic function in* D.

Proof. It follows from (4.1.1) for $v \equiv 1, \Sigma = S$ that

$$\int_G \Delta u\, dx = 0,$$

where $G := \mathrm{int} S$. By virtue of the arbitrariness of G this yields $\Delta u = 0$ in D. □

Theorem 4.1.4. *Let* $u(x)$ *be harmonic in* D *and* $u(x) \in C^1_-(\overline{D})$. *Then*

$$\int_D \left(\sum_{k=1}^{3} \left(\frac{\partial u}{\partial x_k} \right)^2 \right) dx = \int_\Sigma u\frac{\partial u}{\partial n}\, ds. \tag{4.1.3}$$

Relation (4.1.3) is called the *Dirichlet identity*.

Proof. We apply (4.1.1) for the functions $v \equiv 1$ and u^2. Since

$$\frac{\partial(u^2)}{\partial n} = 2u\,\frac{\partial u}{\partial n}, \quad \frac{\partial(u^2)}{\partial x_k} = 2u\,\frac{\partial u}{\partial x_k},$$

$$\frac{\partial^2(u^2)}{\partial x_k^2} = 2\left(\frac{\partial u}{\partial x_k}\right)^2 + 2u\,\frac{\partial^2 u}{\partial x_k^2},$$

we arrive at (4.1.3). □

Theorem 4.1.5. *Let $u(x)$ be harmonic in D and $u(x) \in C^1_-(\overline{D})$. Then*

$$u(x) = \frac{1}{4\pi}\int_\Sigma \left(\frac{1}{r}\frac{\partial u(\xi)}{\partial n_\xi} - u(\xi)\frac{\partial_\xi(\frac{1}{r})}{\partial n_\xi}\right) ds, \quad x \in D, \tag{4.1.4}$$

where ξ is the variable of integration, $r = \|x-\xi\|$, n_ξ is the outer normal to Σ at the point ξ, and ∂_ξ means that the differentiation is performed with respect to ξ (for a fixed x). Formula (4.1.4) is called the basic formula for harmonic functions.

Proof. Fix $x \in D$ and consider the ball $K_\delta(x)$ of radius $\delta > 0$ around the point x. In the domain $D \setminus K_\delta(x)$ we apply Green's formula (4.1.1) for $v = 1/r$. Since $\Delta u = \Delta v = 0$, we have

$$\int_{\Sigma\cup S_\delta(x)} \left(u(\xi)\frac{\partial_\xi(\frac{1}{r})}{\partial n_\xi} - \frac{1}{r}\frac{\partial u(\xi)}{\partial n_\xi}\right) ds = 0,$$

where $S_\delta(x) = \partial K_\delta(x)$ is the sphere. On $S_\delta(x)$:

$$\frac{\partial_\xi(\frac{1}{r})}{\partial n_\xi} = -\frac{\partial(\frac{1}{r})}{\partial r} = \frac{1}{r^2}, \quad r = \delta,$$

and consequently,

$$\begin{aligned} I_\delta &:= \int_{S_\delta(x)} \left(u(\xi)\frac{\partial_\xi(\frac{1}{r})}{\partial n_\xi} - \frac{1}{r}\frac{\partial u(\xi)}{\partial n_\xi}\right) ds \\ &= \int_{S_\delta(x)} \left(u(\xi)\frac{1}{r^2} - \frac{1}{r}\frac{\partial u(\xi)}{\partial n_\xi}\right) ds \\ &= \frac{1}{\delta^2}\int_{S_\delta(x)} u(\xi)\,ds - \frac{1}{\delta}\int_{S_\delta(x)} \frac{\partial u(\xi)}{\partial n_\xi}\,ds. \end{aligned}$$

By Theorem 4.1.2 the second integral is equal to zero, and we obtain

$$\begin{aligned} I_\delta = \frac{1}{\delta^2}\int_{S_\delta(x)} u(\xi)\,ds &= \frac{u(x)}{\delta^2}\int_{S_\delta(x)} ds + \frac{1}{\delta^2}\int_{S_\delta(x)} (u(\xi)-u(x))\,ds \\ &= 4\pi u(x) + J_\delta. \end{aligned}$$

Since the function $u(x)$ is continuous, we have that for each $\varepsilon > 0$ there exists $\delta > 0$ such that $|u(x) - u(\xi)| \leq \varepsilon$ for all $\xi \in \overline{K_\delta(x)}$. Therefore,

$$|J_\delta| \leq \frac{1}{\delta^2} \int_{S_\delta(x)} |u(\xi) - u(x)|\, ds \leq \frac{\varepsilon}{\delta^2} 4\pi\delta^2 = 4\pi\varepsilon.$$

Thus, $I_\delta \to 4\pi u(x)$ as $\delta \to 0$, and we arrive at (4.1.4). □

Theorem 4.1.6. (Mean-Value Theorem). *Let $K_R(x)$ be a ball of radius R around the point x, and let $S_R(x) = \partial K_R(x)$ be the sphere. Suppose that the function $u(\xi)$ is harmonic in $K_R(x)$ and continuous in $\overline{K_R(x)}$. Then*

$$u(x) = \frac{1}{4\pi R^2} \int_{S_R(x)} u(\xi)\, ds. \tag{4.1.5}$$

Proof. We apply the basic formula for harmonic functions (4.1.4) for the ball $K_{\tilde{R}}(x)$ for $\tilde{R} < R$. On $S_{\tilde{R}}$:

$$\frac{\partial_\xi(\frac{1}{r})}{\partial n_\xi} = \frac{\partial(\frac{1}{r})}{\partial r} = -\frac{1}{r^2}, \quad r = \tilde{R},$$

and consequently,

$$\begin{aligned} u(x) &= \frac{1}{4\pi} \int_{S_{\tilde{R}}(x)} \left(\frac{1}{r} \frac{\partial u(\xi)}{\partial n_\xi} + \frac{1}{r^2} u(\xi) \right) ds \\ &= \frac{1}{4\pi\tilde{R}} \int_{S_{\tilde{R}}(x)} \frac{\partial u(\xi)}{\partial n_\xi}\, ds + \frac{1}{4\pi\tilde{R}^2} \int_{S_{\tilde{R}}(x)} u(\xi)\, ds \\ &= \frac{1}{4\pi\tilde{R}^2} \int_{S_{\tilde{R}}(x)} u(\xi)\, ds, \end{aligned}$$

since the first integral here is equal to zero by Theorem 4.1.2. As $\tilde{R} \to R$ we arrive at (4.1.5). □

Theorem 4.1.7 (Maximum Principle). *Let $u(x) \in C(\overline{D})$ and let u be harmonic in D. Then $u(x)$ attains its maximum and minimum on Σ. Moreover, if $u(x) \not\equiv const$, then*

$$\min_{\xi \in \Sigma} u(\xi) < u(x) < \max_{\xi \in \Sigma} u(\xi)$$

for all $x \in D$, i.e. maximum and minimum cannot be attained in D.

Proof. Let $u(x) \not\equiv const$, and let the maximum be attained at some point $x \in D$, i.e. $u(x) = \max_{\xi \in \overline{D}} u(\xi)$. Take a ball $K_R(x)$ such that $\overline{K_R(x)} \subset D$ and such that there exists $\tilde{x} \in S_R(x)$, for which $u(\tilde{x}) < u(x)$. Such choice is possible because $u(\xi) \not\equiv const.$. Since $u(x) > u(\tilde{x})$, there exists $d > 0$ such that $u(x) - u(\tilde{x}) > d > 0$. By virtue of the continuity of the function $u(\xi)$, the inequality also holds in some neighbourhood of $\tilde{x}$. Denote

$$S_R^1(x) = \{\xi \in S_R(x) : u(x) - u(\xi) \geq d > 0\},$$

$$S_R^2(x) = S_R(x) \setminus S_R^1(x).$$

By the mean-value theorem,

$$u(x)=\frac{1}{4\pi R^2}\int_{S_R(x)}u(\xi)\,ds,$$

hence

$$\frac{1}{4\pi R^2}\int_{S_R(x)}(u(x)-u(\xi))\,ds=0.$$

On the other hand,

$$\int_{S_R^1(x)}(u(x)-u(\xi))\,ds\geq dC_0>0,$$

$$\int_{S_R^2(x)}(u(x)-u(\xi))\,ds\geq 0.$$

This contradiction proves the theorem. □

Remark 4.1.3. Let $n=2$, i.e. $x=(x_1,x_2)\in\mathbf{R}^2$. In this case Theorems 4.1.1-4.1.4 and 4.1.7 remain valid (as for any n), and Theorems 4.1.5-4.1.6 require small modifications, namely:

Theorem 4.1.5'. *Let $u(x)$ be harmonic in $D\subset\mathbf{R}^2$ and $u(x)\in C^1_-(\overline{D})$. Then*

$$u(x)=\frac{1}{2\pi}\int_{\Sigma}\left(\left(\ln\frac{1}{r}\right)\frac{\partial u(\xi)}{\partial n_\xi}-u(\xi)\frac{\partial_\xi(\ln\frac{1}{r})}{\partial n_\xi}\right)ds,\ x\in D, \tag{4.1.6}$$

where $r=\|x-\xi\|$, n_ξ is the outer normal to Σ at the point ξ.

Theorem 4.1.6'. *Let $K_R(x)$ be the disc of radius R around the point x, and let $S_R(x)=\partial K_R(x)$ be the circle. Suppose that $u(\xi)$ is harmonic in $K_R(x)$ and continuous in $\overline{K_R(x)}$. Then*

$$u(x)=\frac{1}{2\pi R}\int_{S_R(x)}u(\xi)\,ds. \tag{4.1.7}$$

4.2. Dirichlet and Neumann Problems

Let $D\subset\mathbf{R}^3$ be a bounded domain with the boundary $\Sigma\in PC^1$.

Dirichlet problem

Let a continuous function $\varphi(x)$ be given on Σ. Find a a function $u(x)$ which is harmonic in D and continuous in $\overline{D}$ and has on Σ an assigned value $\varphi(x)$:

$$\left.\begin{aligned}\Delta u=0\quad(x\in D),\\ u_{|\Sigma}=\varphi(x).\end{aligned}\right\} \tag{4.2.1}$$

Theorem 4.2.1. *If a solution of the Dirichlet problem* (4.2.1) *exists, then it is unique.*

Proof. Let $u_1(x)$ and $u_2(x)$ be solutions of problem (4.2.1). Denote $u(x) = u_1(x) - u_2(x)$. Then $u(x) \in C(\overline{D})$, $\Delta u = 0$ in D, and $u_{|\Sigma} = 0$. By the maximum principle, $u(x) \equiv 0$ in $\overline{D}$. □

Definition 4.2.1. The solution of the Dirichlet problem is called stable if for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $|\varphi(x) - \tilde{\varphi}(x)| \leq \delta$ for all $x \in \Sigma$, then $|u(x) - \tilde{u}(x)| \leq \varepsilon$ for all $x \in \overline{D}$. Here u and $\tilde{u}$ are solutions of the Dirichlet problems with the data φ and $\tilde{\varphi}$, respectively.

It follows from the maximum principle that the solution of the Dirichlet problem (4.2.1) is stable (one can take $\delta = \varepsilon$). The question of the existence of the solution of the Dirichlet problem is much more difficult. This question will be studied later by various methods in Sections 4.3-4.6.

Neumann problem

Let a continuous function $\psi(x)$ be given on Σ. Find a function $u(x)$ which is harmonic in D and $u(x) \in C^1_-(\overline{D})$, $\frac{\partial u}{\partial n}{}_{|\Sigma} = \psi$:

$$\left.\begin{aligned} &\Delta u = 0 \ (x \in D), \\ &\left.\frac{\partial u}{\partial n}\right|_{\Sigma} = \psi(x). \end{aligned}\right\} \tag{4.2.2}$$

Theorem 4.2.2. *If a solution of problem* (4.2.2) *exists, then it is unique up to an additive constant.*

Proof. Obviously, if the function $u(x)$ is a solution of problem (4.2.2), then the function $u(x) + C$ is also a solution of (4.2.2). Furthermore, if $u_1(x)$ and $u_2(x)$ are solutions of (4.2.2), then the function $u(x) = u_1(x) - u_2(x)$ has the properties: $\Delta u = 0$ in D, $u(x) \in C^1_-(\overline{D})$ and $\frac{\partial u}{\partial n}{}_{|\Sigma} = 0$. Applying the Dirichlet identity (see Theorem 4.1.4) we obtain

$$\int_D \left(\sum_{k=1}^{3} \left(\frac{\partial u}{\partial x_k} \right)^2 \right) dx = 0,$$

and consequently,

$$\frac{\partial u}{\partial x_k} \equiv 0, \quad k = 1, 2, 3,$$

i.e. $u(x) \equiv const.$. Theorem 4.2.2 is proved. □

Theorem 4.2.3 (Necessary condition for the solvability of the Neumann problem). *If problem* (4.2.2) *has a solution, then*

$$\int_\Sigma \psi(\xi)\, ds = 0.$$

Theorem 4.2.3 is an obvious corollary of Theorem 4.1.2, since, if $u(x)$ is a solution of problem (4.2.2), then

$$\int_{\Sigma} \psi(\xi)\, ds = \int_{\Sigma} \frac{\partial u}{\partial n}\, ds = 0.$$

The Dirichlet and Neumann problems can be considered also in unbounded domains. In this case we need the additional condition $u(\infty)=0$ (for $n=2$ the additional condition has the form $u(x)=O(1),\ |x|\to\infty$). For example, let $D\subset \mathbf{R}^3$ be a bounded set, and $D_1 := \mathbf{R}^3\setminus \overline{D}$. The Dirichlet and Neumann problems in the domain D_1 are called *exterior problems*.

Exterior Dirichlet problem

$$\left.\begin{aligned} \Delta u = 0 \quad (x\in D_1), \\ u_{|\Sigma} = \varphi(x), \quad u(\infty)=0. \end{aligned}\right\} \tag{4.2.3}$$

A function $u(x)$ is called a solution of problem (4.2.3) if $u(x)$ is harmonic in D_1, $u(x)\in C(\overline{D_1})$, $u_{|\Sigma}=\varphi(x)$ and $u(\infty)=0$, i.e.

$$\lim_{R\to\infty} \max_{\|x\|=R} |u(x)| = 0.$$

Exterior Neumann problem

$$\left.\begin{aligned} \Delta u = 0 \quad (x\in D_1), \\ \frac{\partial u}{\partial n}\Big|_{\Sigma} = \psi(x), \quad u(\infty)=0. \end{aligned}\right\} \tag{4.2.4}$$

A function $u(x)$ is called a solution of problem (4.2.4) if $u(x)$ is harmonic in D_1, $u(x)\in C^1_-(\overline{D_1})$, $\dfrac{\partial u}{\partial n}_{|\Sigma} = \psi(x)$ and $u(\infty)=0$.

One can consider the Dirichlet and Neumann problems also in other unbounded regions (a sector, a strip, a half-strip, a half-plane, etc.).

Let us formulate the uniqueness theorem for the exterior Dirichlet problem (4.2.3).

Theorem 4.2.4. *If a solution of problem* (4.2.3) *exists, then it is unique.*

Proof. Let $u_1(x)$ and $u_2(x)$ be solutions of (4.2.3). Denote $u(x)=u_1(x)-u_2(x)$. Then $\Delta u=0$ in D_1, $u(x)\in C(\overline{D_1})$, $u_{|\Sigma}=0$ and $u(\infty)=0$. Let $K_R(0)$ be a ball of radius R around the origin such that $D\subset K_R(0)$. Applying the maximum principle for the region $K_R(0)\setminus D$, we obtain

$$a_R := \min_{\xi\in S_R(0)\cup\Sigma} u(\xi) \le u(x) \le \max_{\xi\in S_R(0)\cup\Sigma} u(\xi) := A_R$$

for all $x\in K_R(0)\setminus D$, where $S_R(0)=\partial K_R(0)$. Since $u(\infty)=0$ and $u_{|\Sigma}=0$, we have $a_R\to 0$ and $A_R\to 0$ as $R\to\infty$. Therefore, $u(x)\equiv 0$, and Theorem 4.2.4 is proved. $\Box$

In order to prove the uniqueness theorem for the exterior Neumann problem we need several assertions which also are of independent interest.

Theorem 4.2.5 (Theorem on a removable singularity). *Let* $0 \in D$, *and let the function* $u(x)$ *be harmonic in* $D \setminus \{0\}$ *and* $\lim\limits_{\|x\| \to 0} \|x\| u(x) = 0$ *(i.e.* $u(x) = o(\frac{1}{\|x\|}), x \to 0$ *). Then the point* $x = 0$ *is a removable singularity for* $u(x)$, *i.e. the function* $u(x)$ *can be defined at the point* 0 *such that* $u(x)$ *becomes harmonic in* D.

Remark 4.2.1. 1) The condition $0 \in D$ is not important and is taken for definiteness; a singularity can be removed similarly at any point.
2) The function $\dfrac{1}{\|x\|}$ is harmonic everywhere except at the point 0 (see Lemma 2.5.1) and it has at 0 an unremovable singularity. This function does not satisfy the last condition of the theorem.

Proof. Let $K_R(0)$ be a ball around the origin such that $\overline{K_R(0)} \subset D$. We take the function $v(x)$ such that $v(x)$ is harmonic in $K_R(0)$, continuous in $\overline{K_R(0)}$ and $v_{|S_R} = u_{|S_R}$, where $S_R = \partial K_R(0)$ is a sphere. In other words, the function $v(x)$ is the solution of the Dirichlet problem for the ball. Below, in Section 4.3 we will prove independently the existence of the solution of the Dirichlet problem for a ball. Therefore, a function $v(x)$ with the above mentioned properties exists and is unique. Denote $w(x) = v(x) - u(x)$. Then $w(x)$ is harmonic in $K_R(0) \setminus \{0\}$ and $w_{|S_R} = 0$. Moreover, since

$$\lim_{\|x\| \to 0} \|x\| u(x) = 0$$

and $v(x)$ is continuous, it follows that

$$\lim_{\|x\| \to 0} \|x\| w(x) = 0.$$

Fix $\alpha > 0$ and consider the function

$$Q_\alpha(x) = \alpha \left(\frac{1}{\|x\|} - \frac{1}{R} \right).$$

Clearly, $Q_\alpha(x)$ is harmonic in $K_R(0) \setminus \{0\}$ and $Q_{\alpha|S_R} = 0$. We have

$$\|x\| (Q_\alpha \pm w(x)) = \alpha - \left(\frac{\alpha \|x\|}{R} \pm \|x\| w(x) \right).$$

There exists $r_\alpha > 0$ such that

$$\left| \frac{\alpha \|x\|}{R} \pm \|x\| w(x) \right| \le \alpha$$

for all $\|x\| \le r_\alpha$. Consider the ring $G_\alpha = \{x : r_\alpha \le x \le R\}$. The functions $w(x)$ and $Q_\alpha(x)$ are harmonic in G_α and on the boundary $|w(x)| \le Q_\alpha(x)$. By the maximum principle this inequality holds also in G_α, i.e.

$$|w(x)| \le \alpha \left(\frac{1}{\|x\|} - \frac{1}{R} \right), \quad x \in G_\alpha.$$

Fix $x \in G_\alpha$. As $\alpha \to 0$ we get $w(x) = 0$. By virtue of the arbitrariness of x we conclude that $w(x) = 0$ for all $x \neq 0$, i.e. $u(x) = v(x)$ for all $x \neq 0$. Defining $u(0) := v(0)$, we arrive at the assertion of Theorem 4.2.5. □

The Kelvin transform

Consider the function $u(x)$ and make the substitution

$$\xi = \frac{x}{\|x\|^2}. \tag{4.2.5}$$

The inverse transform is symmetric:

$$x = \frac{\xi}{\|\xi\|^2}.$$

Denote

$$v(\xi) = \frac{1}{\|\xi\|} u\left(\frac{\xi}{\|\xi\|^2}\right) = \|x\| u(x). \tag{4.2.6}$$

The function $v(\xi)$ is called the *Kelvin transform* for $u(x)$. This transform is symmetric since

$$u(x) = \frac{1}{\|x\|} v\left(\frac{x}{\|x\|^2}\right).$$

Consider a bounded domain $D \in \mathbf{R}^3$ with the boundary Σ, $D_1 = \mathbf{R}^3 \setminus \overline{D}$, and suppose that $0 \in D$. Let D^* be the image of the domain D_1 with respect to the replacement (4.2.5). Then $D_1^* = \mathbf{R}^3 \setminus \overline{D^*}$ and $\Sigma^* = \partial D^*$ are the images of D and Σ, respectively. For example, if $D = K_R(0)$, then $D^* = K_{1/R}(0)$.

Theorem 4.2.6 (Kelvin). *Let $0 \in D$. If the function $u(x)$ is harmonic in D_1, then its Kelvin transform $v(\xi)$ is harmonic in $D^* \setminus \{0\}$. If the function $u(x)$ is harmonic in $D \setminus \{0\}$, then $v(\xi)$ is harmonic in D_1^*.*

Proof. Denote

$$\Delta_x u(x) = \sum_{k=1}^{3} \frac{\partial^2 u}{\partial x_k^2}, \quad \Delta_\xi v(\xi) = \sum_{k=1}^{3} \frac{\partial^2 v}{\partial \xi_k^2}.$$

Using (4.2.5)-(4.2.6) we calculate by differentiation

$$\Delta_x u(x) = \frac{1}{\|x\|^5} \Delta_\xi v(\xi). \tag{4.2.7}$$

All assertions of Theorem 4.2.6 follow from (4.2.7). □

Theorem 4.2.7. *Let the function $u(x)$ be harmonic outside the ball $K_R(0)$ and $u(\infty) = 0$ (i.e. $\lim\limits_{R\to\infty} \max\limits_{\|x\|=R} |u(x)| = 0$). Then*

$$u(x) = O\left(\frac{1}{\|x\|}\right), \quad \frac{\partial u}{\partial x_k} = O\left(\frac{1}{\|x\|^2}\right), \quad x \to \infty.$$

Proof. Let
$$v(\xi) = \frac{1}{\|\xi\|}\, u\left(\frac{\xi}{\|\xi\|^2}\right)$$
be the Kelvin transform for $u(x)$. By Theorem 4.2.6, the function $v(\xi)$ is harmonic in the domain $K_{1/R}(0)\setminus\{0\}$. Since $u(\infty)=0$, we have $\|\xi\|v(\xi)\to 0$ as $\xi\to 0$. Then, by Theorem 4.2.5, the function $v(\xi)$ is harmonic in $K_{1/R}(0)$. In particular, this yields
$$v(\xi)=O(1), \quad \frac{\partial v}{\partial \xi_k}=O(1) \quad \text{for} \quad \xi\to 0. \tag{4.2.8}$$
Since
$$u(x)=\frac{1}{\|x\|}\, v(\xi),$$
we have, by virtue of (4.2.8), that
$$u(x)=O\left(\frac{1}{\|x\|}\right), \quad x\to\infty.$$
Furthermore,
$$\frac{\partial u}{\partial x_i}=\frac{\partial}{\partial x_i}\left(\frac{1}{\|x\|}\, v(\xi)\right)=-\frac{x_i}{\|x\|^3}\, v(\xi)+\frac{1}{\|x\|}\sum_{k=1}^{3}\frac{\partial v}{\partial \xi_k}\frac{\partial \xi_k}{\partial x_i}.$$
Since
$$\frac{\partial \xi_k}{\partial x_i}=\frac{\delta_{ki}}{\|x\|^2}+x_k\frac{\partial}{\partial x_i}\left(\frac{1}{\|x\|^2}\right),$$
$$\frac{\partial}{\partial x_i}\left(\frac{1}{\|x\|^2}\right)=-\frac{2x_i}{\|x\|^4},$$
where δ_{ki} is the Kronecker delta, one gets
$$\frac{\partial u}{\partial x_i}=-\frac{x_i}{\|x\|^3}\, v(\xi)+\frac{1}{\|x\|^3}\frac{\partial v}{\partial \xi_i}-\frac{2x_i}{\|x\|^5}\sum_{k=1}^{3}x_k\frac{\partial v}{\partial \xi_k}.$$
In view of (4.2.8) this yields
$$\frac{\partial u}{\partial x_i}=O\left(\frac{1}{\|x\|^2}\right), \quad x\to\infty.$$
Theorem 4.2.7 is proved. □

Let us now prove the uniqueness theorem for the exterior Neumann problem (4.2.4).

Theorem 4.2.8. *If the solution of problem* (4.2.4) *exists, then it is unique.*

Proof. Let $u_1(x)$ and $u_2(x)$ be solutions of problem (4.2.4). Denote $u(x)=u_1(x)-u_2(x)$. Then the function $u(x)$ is harmonic in D_1, $u(\infty)=0$ and $\frac{\partial u}{\partial n}_{|\Sigma}=0$. Let $K_R(0)$ be the ball of radius R around the origin such that $D \subset K_R(0)$. We apply the Dirichlet identity to the domain $K_R(0)\setminus D$:

$$\int_{K_R(0)\setminus D} \left(\sum_{k=1}^{3} \left(\frac{\partial u}{\partial x_k} \right)^2 \right) dx = \int_{S_R} u \frac{\partial u}{\partial n} ds, \tag{4.2.9}$$

where $S_R = \partial K_R(0)$. By Theorem 4.2.7, on S_R we have

$$|u(x)| \le \frac{C}{R}, \quad |\frac{\partial u}{\partial n}| \le \frac{C}{R^2},$$

and consequently,

$$\left| \int_{S_R} u \frac{\partial u}{\partial n} ds \right| \le \frac{C}{R} \to 0 \quad \text{as} \quad R \to \infty.$$

Then from (4.2.9) as $R \to \infty$ we obtain

$$\int_{D_1} \left(\sum_{k=1}^{3} \left(\frac{\partial u}{\partial x_k} \right)^2 \right) dx = 0.$$

Therefore,

$$\frac{\partial u}{\partial x_k} \equiv 0, \quad k = 1,2,3,$$

i.e. $u(x) \equiv const.$. Since $u(\infty) = 0$, we conclude that $u(x) \equiv 0$, and Theorem 4.2.8 is proved. □

Reduction of the exterior Dirichlet problem to the interior one

Let $0 \in D$. Let $u(x)$ be the solution of problem (4.2.3) and let $v(\xi)$ be the Kelvin transform for $u(x)$. Then, by Theorems 4.2.5-4.2.6, the function $v(\xi)$ is harmonic in D^*. Moreover, $v(\xi) \in C(\overline{D^*})$ and $v_{|\Sigma^*} = \varphi^*$, where φ^* is the Kelvin transform for φ. Thus, $v(\xi)$ is the solution of the Dirichlet problem for the bounded domain D^*.

4.3. The Green's Function Method

1. Let $D \subset \mathbf{R}^3$ be a bounded domain with the boundary $\Sigma \in PC^1$. We consider the Dirichlet problem

$$\left. \begin{aligned} \Delta u = 0 \quad (x \in D), \\ u_{|\Sigma} = \varphi(x), \end{aligned} \right\} \tag{4.3.1}$$

where $\varphi(x) \in C(\Sigma)$. Let $u(x)$ be the solution of problem (4.3.1), and let $v(x)$ be a harmonic function in D. Moreover, suppose that $u(x), v(x) \in C^1_-(\overline{D})$. Then, by virtue of Theorem 4.1.1 and 4.1.5, we obtain

$$u(x) = \frac{1}{4\pi} \int_{\Sigma} \left(\frac{1}{r} \frac{\partial u(\xi)}{\partial n_\xi} - u(\xi) \frac{\partial_\xi (\frac{1}{r})}{\partial n_\xi} \right) ds, \; x \in D, \; r = \|x - \xi\|,$$

$$0 = \int_{\Sigma} \left(v(\xi) \frac{\partial u(\xi)}{\partial n_{\xi}} - u(\xi) \frac{\partial v(\xi)}{\partial n_{\xi}} \right) ds.$$

Therefore,

$$u(x) = \frac{1}{4\pi} \int_{\Sigma} \left(\left(v + \frac{1}{r} \right) \frac{\partial u}{\partial n} - u \frac{\partial}{\partial n} \left(v + \frac{1}{r} \right) \right) ds. \tag{4.3.2}$$

We require that $(v + \frac{1}{r})_{|\Sigma} = 0$. Then for finding the function $v = v(\xi, x)$ we obtain the following Dirichlet problem with respect to ξ for a fixed $x \in D$:

$$\left. \begin{aligned} \Delta_{\xi} v &= 0 \quad (\xi \in D), \\ v_{|\Sigma} &= -\frac{1}{r}. \end{aligned} \right\} \tag{4.3.3}$$

Consider the function

$$G(x, \xi) = v + \frac{1}{r},$$

where v is the solution of problem (4.3.3). The function $G(x, \xi)$ is called the *Green's function*. Formula (4.3.2) takes the form

$$u(x) = -\frac{1}{4\pi} \int_{\Sigma} \varphi(\xi) \frac{\partial_{\xi} G(x, \xi)}{\partial n_{\xi}} ds, \quad x \in D. \tag{4.3.4}$$

Thus, if the solutions of the problems (4.3.1) and (4.3.3) exist (and have on Σ normal derivatives), then formula (4.3.4) is valid.

Remark 4.3.1. In order to construct the solution of the Dirichlet problem (4.3.1) by (4.3.4) we first need to solve the Dirichlet problem (4.3.3). For some domains the Dirichlet problem (4.3.3) can be solved explicitly. Below, using the Green's function method we will construct the solution of the Dirichlet problem for a ball.

By similar arguments one can solve also the Neumann problem

$$\Delta u = 0 \quad (x \in D),$$

$$\left. \frac{\partial u}{\partial n} \right|_{\Sigma} = \psi(x).$$

The solution has the form

$$u(x) = \frac{1}{4\pi} \int_{\Sigma} \psi(\xi) \, \tilde{G}(x, \xi) \, ds, \quad x \in D,$$

where

$$\tilde{G}(x, \xi) = \tilde{v} + \frac{1}{r},$$

and the function $\tilde{v}(\xi, x)$ is the solution of the following Neumann problem with respect to ξ:

$$\Delta_{\xi} \tilde{v} = 0,$$

$$\left. \frac{\partial_{\xi} \tilde{v}}{\partial n_{\xi}} \right|_{\Sigma} = -\frac{\partial_{\xi}(1/r)}{\partial n_{\xi}}.$$

Below we will need the following auxiliary assertion.

Lemma 4.3.1. *Let* l *be a vector and* $r=\|\xi-x\|$. *Then*

$$\frac{\partial_\xi r}{\partial l}=\cos(l,\bar r),$$

where $\bar r=(\xi_1-x_1,\xi_2-x_2,\xi_3-x_3)$. *Symmetrically,*

$$\frac{\partial_x r}{\partial l}=-\cos(l,\bar r).$$

Indeed,

$$\begin{aligned}\frac{\partial_\xi r}{\partial l}&=\sum_{k=1}^{3}\frac{\partial r}{\partial \xi_k}\cos(l,\xi_k)=\sum_{k=1}^{3}\frac{\xi_k-x_k}{r}\cos(l,\bar\xi_k)\\&=\sum_{k=1}^{3}\cos(\bar r,\xi_k)\cos(l,\xi_k)=(l^0,\bar r^0)\\&=\|l^0\|\cdot\|\bar r^0\|\cos(l,\bar r)=\cos(l,\bar r),\end{aligned}$$

where

$$\begin{aligned}l^0&=(\cos(l,\xi_1),\cos(l,\xi_2),\cos(l,\xi_3)),\\ \bar r^0&=(\cos(\bar r,\xi_1),\cos(\bar r,\xi_2),\cos(\bar r,\xi_3))\end{aligned}$$

are unit vectors for l and $\bar r$, respectively, and $(l^0,\bar r^0)$ is the scalar product of the vectors l^0 and $\bar r^0$.

2. Solution of the Dirichlet problem for a ball

Let K_R be a ball of radius R around the point x^0, and let $S_R=\partial K_R$ be the sphere. We consider the Dirichlet problem

$$\left.\begin{aligned}\Delta u=0 &\quad (x\in K_R),\\ u_{|S_R}=\varphi(x), &\quad \varphi(x)\in C(S_R).\end{aligned}\right\}\tag{4.3.5}$$

Let $x\in K_R$. On the ray x^0x choose the point x^1 such that

$$\|x^0-x\|\cdot\|x^0-x^1\|=R^2.$$

Let $\xi\in S_R$. Denote

$$r=\|\xi-x\|,\ \rho=\|x^0-x\|,\ r_1=\|\xi-x^1\|,\ \rho_1=\|x^0-x^1\|$$

(see fig. 4.3.1).

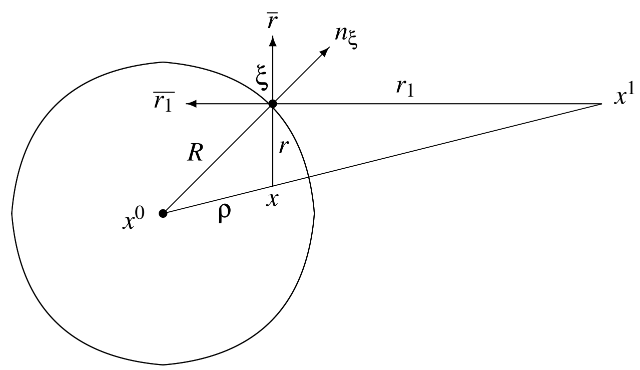

Figure 4.3.1.

We note that the triangles $\Delta_{x^0\xi x}$ and $\Delta_{x^0x^1\xi}$ are similar since they have a common angle and

$$\frac{\rho}{R} = \frac{R}{\rho_1}.$$

Hence

$$\frac{\rho}{R} = \frac{R}{\rho_1} = \frac{r}{r_1}, \quad \xi \in S_R. \tag{4.3.6}$$

In our case the Dirichlet problem (4.3.3) has the form

$$\left.\begin{aligned} \Delta_\xi v = 0 \quad (\xi \in K_R), \\ v_{|S_R} = -\frac{1}{r}. \end{aligned}\right\} \tag{4.3.7}$$

Let us show that the solution of problem (4.3.7) has the form

$$v(\xi, x) = \frac{\alpha}{r_1}, \quad \alpha - const.$$

The function $v(\xi, x)$ is harmonic with respect to ξ everywhere except at the point x^1, and in particular, it is harmonic in $\overline{K_R}$. The boundary condition yields

$$\frac{\alpha}{r_1} = -\frac{1}{r}, \quad \xi \in S_R.$$

Using (4.3.6) we calculate

$$\alpha = -\frac{R}{\rho}.$$

Thus,

$$v(\xi, x) = -\frac{R}{\rho r_1},$$

and consequently, the Green's function has the form

$$G(x, \xi) = \frac{1}{r} - \frac{R}{\rho r_1}. \tag{4.3.8}$$

Let us show that

$$\frac{\partial_\xi G(x,\xi)}{\partial n_\xi} = \frac{\rho^2 - R^2}{Rr^3}. \tag{4.3.9}$$

Indeed, by virtue of Lemma 4.3.1,

$$\frac{\partial_\xi r}{\partial n_\xi} = \cos(n_\xi, \bar{r}), \quad \frac{\partial_\xi r_1}{\partial n_\xi} = \cos(n_\xi, \bar{r}_1). \tag{4.3.10}$$

Using the cosine theorem we calculate

$$\cos(n_\xi, \bar{r}) = \frac{R^2 + r^2 - \rho^2}{2Rr}, \quad \cos(n_\xi, \bar{r}_1) = \frac{R^2 + r_1^2 - \rho_1^2}{2Rr_1}. \tag{4.3.11}$$

Differentiating (4.3.8) and using (4.3.6), (4.3.10) and (4.3.11) we get

$$\frac{\partial_\xi G(x,\xi)}{\partial n_\xi} = \frac{\partial_\xi(\frac{1}{r})}{\partial n_\xi} - \frac{R}{\rho}\frac{\partial_\xi(\frac{1}{r_1})}{\partial n_\xi} = -\frac{\cos(\bar{n}_\xi, \bar{r})}{r^2} + \frac{R}{\rho}\frac{\cos(\bar{n}_\xi, \bar{r}_1)}{r_1^2}$$

$$= \frac{\rho^2 - R^2 - r^2}{2Rr^3} + \frac{R^2 + r_1^2 - \rho_1^2}{2\rho r_1^3} = \frac{\rho^2 - R^2}{2Rr^3} - \frac{1}{2Rr} + \frac{R^2 - \rho_1^2}{2\rho r_1^3} + \frac{1}{2\rho r_1}$$

$$= \frac{\rho^2 - R^2}{2Rr^3} + \frac{R^2\rho^2 - (\rho_1\rho)^2}{2(\rho r_1)^3} = \frac{\rho^2 - R^2}{2Rr^3} + \frac{\rho^2 - R^2}{2Rr^3} = \frac{\rho^2 - R^2}{Rr^3},$$

i.e. (4.3.9) is valid. Substituting now (4.3.9) into (4.3.4) we arrive at the formula

$$u(x) = \frac{1}{4\pi}\int_{S_R} \varphi(\xi)\,\frac{R^2 - \rho^2}{Rr^3}\,ds, \quad x \in K_R, \tag{4.3.12}$$

which is called the *Poisson formula*. Thus, we have proved that if the solution of problem (4.3.5) exists, then it is given by (4.3.12).

Theorem 4.3.1. *For each continuous function* φ *on* S_R *the solution of the Dirichlet problem* (4.3.5) *for the ball exists, is unique and is represented by formula* (4.3.12).

Proof. It is sufficient to prove that the function $u(x)$, defined by (4.3.12), is a solution of problem (4.3.5). First we will prove that $u(x)$ is harmonic in K_R. Let $\overline{G} \subset K_R$ be a bounded closed domain. Then for all $x \in \overline{G}$, $\xi \in S_R$ we have $r = \|x - \xi\| \ge d > 0$, where d is the distance from $\overline{G}$ to S_R. Therefore, $u(x) \in C^\infty(\overline{G})$, and all derivatives of $u(x)$ can be obtained by differentiation under the integral sign. In particular,

$$\Delta u = \frac{1}{4\pi}\int_{S_R} \varphi(\xi)\Delta\left(\frac{R^2 - \rho^2}{Rr^3}\right) ds.$$

Using (4.3.10)-(4.3.11) we calculate

Clearly,

$$\Delta\left(\frac{R^2 - \rho^2}{Rr^3}\right) = \Delta\left(\frac{R^2 - \rho^2 + r^2}{Rr^3}\right) - \Delta\left(\frac{1}{Rr}\right) = \Delta\left(\frac{R^2 - \rho^2 + r^2}{Rr^3}\right).$$

Using (4.3.10)-(4.3.11) we calculate

$$\Delta\left(\frac{R^2-\rho^2}{Rr^3}\right)=2\Delta\left(\frac{\cos(n_\xi,\bar{r})}{r^2}\right)$$

$$=-2\Delta\left(\frac{\partial_\xi(\frac{1}{r})}{\partial n_\xi}\right)=-2\frac{\partial_\xi}{\partial n_\xi}\left(\Delta\left(\frac{1}{r}\right)\right)=0,$$

and consequently, $\Delta u(x)=0$ in $\overline{G}$. By virtue of the arbitrariness of $\overline{G}$ we conclude that $u(x)\in C^\infty(K_R)$ and $u(x)$ is harmonic in K_R.

Fix $x^*\in\Sigma$. Let us show that

$$\lim_{x\to x^*,x\in K_R}u(x)=\varphi(x^*). \tag{4.3.13}$$

For this purpose we consider the auxiliary Dirichlet problem

$$\left.\begin{aligned}\Delta w=0 \quad (x\in K_R),\\ w_{|S_R}=1.\end{aligned}\right\} \tag{4.3.14}$$

Obviously, problem (4.3.14) has the unique solution $w(x)\equiv 1$. Hence, in view of (4.3.12),

$$1=\frac{1}{4\pi}\int_{S_R}\frac{R^2-\rho^2}{Rr^3}\,ds, \tag{4.3.15}$$

and consequently,

$$u(x)-\varphi(x^*)=\frac{1}{4\pi R}\int_{S_R}\Big(\varphi(\xi)-\varphi(x^*\Big)\frac{R^2-\rho^2}{r^3}\,ds,\quad x\in K_R. \tag{4.3.16}$$

Fix $\varepsilon>0$. Since the function φ is continuous, there exists $\delta>0$ such that for $\|\xi-x^*\|\le\delta$ one has $|\varphi(\xi)-\varphi(x^*)|\le\varepsilon/2$. Let $x\in K_R$, $\|x-x^*\|\le\delta/2$. Denote

$$S_R^1=\{\xi\in S_R:\ \|\xi-x^*\|\le\delta\},\quad S_R^2=S_R\setminus S_R^1.$$

Clearly, $r\ge\delta/2$ on S_R^2. Let $C=\max_\xi|\varphi(\xi)|$. Then it follows from (4.3.15)-(4.3.16) that for $x\in K_R$, $\|x-x^*\|\le\delta/2$,

$$|u(x)-\varphi(x^*)|\le\frac{\varepsilon}{2}\cdot\frac{1}{4\pi R}\int_{S_R^1}\frac{R^2-\rho^2}{r^3}\,ds+\frac{2C}{4\pi R}\cdot\frac{R^2-\rho^2}{(\delta/2)^3}\int_{S_R^2}ds$$

$$\le\frac{\varepsilon}{2}+\frac{2^4CR}{\delta^3}(R^2-\rho^2).$$

Let now $x\to x^*$. Then $\rho^2\to R^2$, i.e. there exists δ_1 such that for $\|x-x^*\|\le\delta_1$ one has

$$\frac{2^4CR}{\delta^3}(R^2-\rho^2)\le\frac{\varepsilon}{2}.$$

Thus, we have proved that for each $\varepsilon>0$ there exists $\delta_1>0$ such that if $\|x-x^*\|\le\delta_1$, then $|u(x)-\varphi(x^*)|\le\varepsilon$, i.e. (4.3.15) is valid. Therefore, $u(x)\in C(\overline{K_R})$ and $u_{|S_R}=\varphi$. Theorem 4.3.1 is proved. □

3. Two-dimensional case. Solution of the Dirichlet problem for a disc

In the case $n=2$, $x=(x_1,x_2)\in\mathbf{R}^2$, there are similar results to those provided above. Let us briefly formulate them. The solution of the Dirichlet problem (4.3.1) for $n=2$ is given by the formula

$$u(x)=-\frac{1}{2\pi}\int_{\Sigma}\varphi(\xi)\,\frac{\partial_\xi G(x,\xi)}{\partial n_\xi}\,ds,\quad x\in D, \tag{4.3.17}$$

where

$$G(x,\xi)=v+\ln\frac{1}{r}$$

is Green's function, and $v=v(\xi,x)$ is the solution of the following Dirichlet problem with respect to ξ:

$$\left.\begin{aligned}\Delta_\xi v=0\quad(\xi\in D),\\ v_{|\Sigma}=-\ln\frac{1}{r}.\end{aligned}\right\} \tag{4.3.18}$$

In the case of a disc (i.e. when $D=K_R$), Green's function has the form

$$G(x,\xi)=\ln\frac{1}{r}-\ln\frac{R}{\rho r_1}$$

and

$$\frac{\partial_\xi G(x,\xi)}{\partial n_\xi}=\frac{\rho^2-R^2}{Rr^2}.$$

The solution of the Dirichlet problem (4.3.5) for $n=2$ exists, is unique and is given by the formula

$$u(x)=\frac{1}{2\pi}\int_{S_R}\varphi(\xi)\,\frac{R^2-\rho^2}{Rr^2}\,ds,\quad x\in K_R,\ S_R=\partial K_R, \tag{4.3.19}$$

which is called the *Poisson formula for a disc*. We transform (4.3.19) with the help of polar coordinates. We introduce the polar coordinate system with the pole at the point x^0 and with an arbitrary axis. Let $\arg x=\alpha$, $\arg\xi=\theta$, i.e. (ρ,α) are the polar coordinates of x, and (R,θ) are the polar coordinates of $\xi\in S_R$. Then

$$\varphi(\xi)=\varphi(R\cos\theta,R\sin\theta):=\tilde\varphi(\theta),$$

$$u(x)=u(\rho\cos\alpha,\rho\sin\alpha):=\tilde u(\rho,\alpha),$$

and by the cosine theorem, $r^2=R^2+\rho^2-2R\rho\cos(\alpha-\theta)$. Therefore, (4.3.19) takes the form

$$\tilde u(\rho,\alpha)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{R^2-\rho^2}{R^2+\rho^2-2R\rho\cos(\alpha-\theta)}\,\tilde\varphi(\theta)\,d\theta. \tag{4.3.20}$$

Let us show that formula (4.3.20) can be also obtained by the method of separation of variables. Let for simplicity, $x^0=0$. The Laplace operator in polar coordinates has the form

$$\Delta u=\frac{1}{\rho}(\rho u_\rho)_\rho+\frac{1}{\rho^2}u_{\alpha\alpha},$$

where $x_1 = \rho\cos\alpha$, $x_2 = \rho\sin\alpha$ (see [1, Chapter 4]). Thus, we obtain the following Dirichlet problem for a disc with respect to the function $u(\rho,\alpha)$:

$$\frac{1}{\rho}(\rho u_\rho)_\rho + \frac{1}{\rho^2}u_{\alpha\alpha} = 0, \quad u_{|\rho=R} = \varphi(\alpha), \tag{4.3.21}$$

where $\varphi(\alpha)$ is continuous and $\varphi(\alpha) = \varphi(2\pi+\alpha)$. We expand $\varphi(\alpha)$ into a Fourier series:

$$\varphi(\alpha) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos n\alpha + b_n\sin n\alpha), \tag{4.3.22}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\varphi(\alpha)\cos n\alpha\, d\alpha, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\varphi(\alpha)\sin n\alpha\, d\alpha.$$

We seek particular solutions of equation (4.3.21) of the form

$$u(\rho,\alpha) = V(\rho)w(\alpha).$$

Then

$$\frac{\rho(\rho V'(\rho))'}{V(\rho)} = -\frac{w''(\alpha)}{w(\alpha)} = \lambda^2,$$

and consequently,

$$\left.\begin{array}{c} \rho(\rho V'(\rho))' = \lambda^2 V(\rho), \\ w''(\alpha) + \lambda^2 w(\alpha) = 0,\ w(\alpha+2\pi) = w(\alpha). \end{array}\right\} \tag{4.3.23}$$

The general solutions of the equations (4.3.23) are:

$$w(\alpha) = A\cos\lambda\alpha + B\sin\lambda\alpha,$$
$$V(\rho) = C\rho^{\lambda} + D\rho^{-\lambda}.$$

For $\lambda = n$ we take

$$w_n(\alpha) = A_n\cos n\alpha + B_n\sin n\alpha,$$
$$V_n(\rho) = \rho^n,$$

hence

$$u_n(\rho,\alpha) = \rho^n(A_n\cos n\alpha + B_n\sin n\alpha),\ \ n \geq 0.$$

We seek the solution of problem (4.3.21) in the form

$$u(\rho,\alpha) = \sum_{n=0}^{\infty}\rho^n(A_n\cos n\alpha + B_n\sin n\alpha). \tag{4.3.24}$$

The boundary condition for $\rho = R$ yields

$$\varphi(\alpha) = \sum_{n=0}^{\infty}R^n(A_n\cos n\alpha + B_n\sin n\alpha).$$

Comparing this relation with (4.3.22) we calculate

$$A_0 = \frac{a_0}{2}, \quad A_n = \frac{a_n}{R^n}, \quad B_n = \frac{b_n}{R^n}, \quad n \geq 1.$$

Thus, formula (4.3.24) takes the form

$$u(\rho,\alpha) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \left(\frac{\rho}{R}\right)^n (a_n \cos n\alpha + b_n \sin n\alpha). \tag{4.3.25}$$

We substitute the expressions for a_n and b_n into (4.3.25):

$$u(\rho,\alpha) = \frac{1}{\pi}\int_{-\pi}^{\pi} \varphi(\theta) \left(\frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{\rho}{R}\right)^n (\cos n\theta \cos n\alpha + \sin n\theta \sin n\alpha)\right) d\theta$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \varphi(\theta) \left(\frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{\rho}{R}\right)^n \cos n(\theta - \alpha)\right) d\theta.$$

Denote $t = \dfrac{\rho}{R} < 1.$ We have

$$\frac{1}{2} + \sum_{n=1}^{\infty} t^n \cos n(\alpha - \theta)$$

$$= \frac{1}{2}\left(1 + \sum_{n=1}^{\infty} \left((te^{i(\alpha-\theta)})^n + (te^{-i(\alpha-\theta)})^n\right)\right)$$

$$= \frac{1}{2}\left(1 + \frac{te^{i(\alpha-\theta)}}{1 - te^{i(\alpha-\theta)}} + \frac{te^{-i(\alpha-\theta)}}{1 - te^{-i(\alpha-\theta)}}\right)$$

$$= \frac{1}{2} \cdot \frac{1 - t^2}{1 - 2t\cos(\alpha - \theta) + t^2}.$$

Thus, (4.3.25) takes the form

$$u(\rho,\alpha) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{R^2 - \rho^2}{R^2 + \rho^2 - 2R\rho\cos(\alpha - \theta)} \varphi(\theta)\, d\theta,$$

which coincides with (4.3.20).

4.4. The Method of Upper and Lower Functions

Let for definiteness $n = 3,$ and let $D \subset \mathbf{R}^3$ be a bounded domain with the boundary $\Sigma \in PC^1$. In this section we consider a method for the solution of the Dirichlet problem:

$$\left.\begin{aligned} \Delta u = 0 \quad (x \in D), \\ u_{|\Sigma} = \varphi, \quad \varphi \in C(\Sigma). \end{aligned}\right\} \tag{4.4.1}$$

First we prove several assertions which are also of independent interest.

Harnack's inequality. *Let the function* $u(x)$ *be harmonic in the ball* $K_R = \{x : \|x - x^0\| < R\}$, *continuous in* $\overline{K_R}$ *and* $u(x) \geq 0$. *Then*

$$\frac{R(R-\rho)}{(R+\rho)^2}\,u(x^0) \leq u(x) \leq \frac{R(R+\rho)}{(R-\rho)^2}\,u(x^0),\ x \in K_R, \tag{4.4.2}$$

where $\rho = \|x - x^0\|$.

Proof. We will use the Poisson formula (4.3.12) for a ball. Since $R - \rho \leq r \leq R + \rho$ (see fig. 4.3.1), we have

$$\frac{1}{4\pi R}\cdot\frac{R^2-\rho^2}{(R+\rho)^3}\int_{S_R} u(\xi)\,ds \leq u(x) \leq \frac{1}{4\pi R}\cdot\frac{R^2-\rho^2}{(R-\rho)^3}\int_{S_R} u(\xi)\,ds,\ x \in K_R.$$

Applying the mean-value theorem we arrive at (4.4.2). □

Theorem 4.4.1 (The first Harnack theorem). *Let the functions* $u_n(x)$, $n \geq 1$ *be harmonic in* D, *continuous in* $\overline{D}$ *and let* $\{u_n(x)\}_{n\geq 1}$ *converge uniformly on* Σ. *Then* $\{u_n(x)\}_{n\geq 1}$ *converges uniformly in* $\overline{D}$, *and the limit function* $u(x)$ *is harmonic in* D *and continuous in* $\overline{D}$.

Proof. Since $\{u_n(x)\}_{n\geq 1}$ converges uniformly on Σ, for each $\varepsilon > 0$ there exists N such that $|u_{n+p}(x) - u_n(x)| \leq \varepsilon$ for all $n \geq N, p \geq 0, x \in \Sigma$. By the maximum principle, this inequality remains valid also for all $x \in \overline{D}$. Hence $\{u_n(x)\}_{n\geq 1}$ converges uniformly in $\overline{D}$, and the limit function $u(x)$ is continuous in $\overline{D}$. Let us show that $u(x)$ is harmonic in D. For this purpose it is sufficient to prove that the function $u(x)$ is harmonic in each ball $K_R(x^0) \subset D$. Since the functions $u_n(x)$ are harmonic in D and continuous in $\overline{D}$, we have

$$u_n(x) = \frac{1}{4\pi}\int_{S_R} u_n(\xi)\,\frac{R^2-\rho^2}{Rr^3}\,ds, \quad x \in K_R(x^0), \quad S_R = \partial K_R(x^0).$$

As $n \to \infty$ we get

$$u(x) = \frac{1}{4\pi}\int_{S_R} u(\xi)\,\frac{R^2-\rho^2}{Rr^3}\,ds, \quad x \in K_R(x^0),$$

and consequently, the function $u(x)$ is harmonic in $K_R(x^0)$. Theorem 4.4.1 is proved. □

Theorem 4.4.2 (The second Harnack theorem). *Let the functions* $u_n(x)$, $n \geq 1$ *be harmonic in* D, *and let the sequence* $\{u_n(x)\}_{n\geq 1}$ *be monotone in* D *and convergent at least in one point* $x^0 \in D$. *Then* $\{u_n(x)\}_{n\geq 1}$ *converges in* D *to a harmonic function* $u(x)$, *and the convergence is locally uniform inside* D *(i.e. it is uniform in each bounded closed domain* $\overline{G} \subset D$*).*

Proof. 1) Let for definiteness, the sequence be non-increasing: $u_{n+1}(x) \leq u_n(x)$. Let us show that $\{u_n(x)\}_{n\geq 1}$ converges uniformly in the ball $\overline{K_R(x^0)} \subset D$. For this purpose we take the ball $K_{R_1}(x^0)$, $R_1 > R$ such that $\overline{K_{R_1}(x^0)} \subset D$. Using the Harnack inequality for the ball $K_{R_1}(x^0)$ and for the function $u_n(x) - u_{n+p}(x)$, we obtain

$$0 \leq u_n(x) - u_{n+p}(x) \leq \frac{R_1(R_1+\rho)}{(R_1-\rho)^2}\Big(u_n(x^0) - u_{n+p}(x^0)\Big),\ x \in K_{R_1}(x^0).$$

Let now $x \in \overline{K_R(x^0)}$. Then $\rho \le R$, and consequently,

$$0 \le u_n(x) - u_{n+p}(x) \le \frac{R_1(R_1+R)}{(R_1-R)^2}\Big(u_n(x^0) - u_{n+p}(x^0)\Big),\ x \in K_R(x^0). \tag{4.4.3}$$

Since $\{u_n(x^0)\}$ converges, we have that for each $\varepsilon > 0$ there exists N such that

$$0 \le u_n(x^0) - u_{n+p}(x^0) \le \varepsilon$$

for all $n \ge N$, $p \ge 0$. Then we infer from (4.4.3) that

$$0 \le u_n(x) - u_{n+p}(x) \le \frac{R_1(R_1+R)}{(R_1-R)^2}\,\varepsilon := \varepsilon_1.$$

Therefore, the sequence $\{u_n(x)\}$ converges uniformly in $\overline{K_R(x^0)}$. By Theorem 4.4.1, the limit function $u(x)$ is harmonic in $K_R(x^0)$ and continuous in $\overline{K_R(x^0)}$.

2) Let us show that the sequence $\{u_n(x)\}$ converges uniformly in each closed ball $\overline{K_{R^*}(x^*)} \subset D$. We connect the points x^0 and x^* by a curve lying in D. One can construct a finite number of balls $K_j = K_{R_j}(x^j)$, $j = \overline{0,m}$ such that $\overline{K_j} \subset D$, $K_0 = K_R(x^0)$, $K_m = K_{R^*}(x^*)$ and $x^j \in K_{j-1}$ (i.e. the center of the next ball lies in the previous one). Using the first part of the proof we get by induction that $\{u_n(x)\}$ converges uniformly in $\overline{K_j}$ for $j = \overline{0,m}$, and in particular, in $\overline{K_{R^*}(x^*)}$.

3) By the Heine-Borel lemma, for each bounded closed domain $\overline{G} \subset D$, there exists a finite number of balls covering $\overline{G}$. In each of these balls the sequence $\{u_n(x)\}$ converges uniformly; hence $\{u_n(x)\}$ converges uniformly in $\overline{G}$. □

Definition 4.4.1. Let $v(x) \in C(\overline{D})$, and let $K = K_R(x^0)$ be a ball such that $K \subset D$. The function

$$(v)_K(x) = \begin{cases} v(x), & x \notin K, \\ \dfrac{1}{4\pi R}\displaystyle\int_{S_R} v(\xi)\,\frac{R^2-\rho^2}{r^3}\,ds, & x \in K, \end{cases}$$

is called *the cut-off function* of $v(x)$ with respect to the ball K. Here, as in Section 4.3, $r = \|x-\xi\|$, $\rho = \|x-x^0\|$ and $S_R = \partial K$ is a sphere. In other words, the cut-off function $(v)_K$ changes the function v in the ball K into the harmonic function (the solution of the Dirichlet problem for the ball), and leaves v fixed outside K. Therefore, $(v)_K$ is harmonic in K and continuous in $\overline{D}$.

Definition 4.4.2. 1) The function $v(x)$ is called *superharmonic* in D, if $v(x) \in C(\overline{D})$ and $(v)_K(x) \le v(x)$ for each ball $K \subset D$.
2) The function $v(x)$ is called *subharmonic* in D, if $v(x) \in C(\overline{D})$ and $(v)_K(x) \ge v(x)$ for each ball $K \subset D$.

Definition 4.4.3. Let on Σ a continuous function $\varphi(x)$ be given.
1) The function $v(x)$ is called an *upper function* (for φ), if $v(x)$ is superharmonic in D and $v_{|\Sigma} \ge \varphi$.
2) The function $v(x)$ is called a *lower function* (for φ), if $v(x)$ is subharmonic in D and $v_{|\Sigma} \le \varphi$.

All statements concerning upper and lower functions in this section are related to the same function φ – therefore we omit below the appendix “for φ”.

The idea of the method of upper and lower functions is to obtain the solution of the Dirichlet problem (4.4.1) as the infimum of the upper functions (or as the supremum of the lower functions). In order to realize this idea we need to study the properties of the upper and lower functions.

Theorem 4.4.3. *Let $u(x)$ be harmonic in D and continuous in $\overline{D}$. Then $u(x)$ is superharmonic and subharmonic in D simultaneously.*

Indeed, since $u(x)$ is harmonic in D, we have $(u)_K = u$ for each ball $K \subset D$. Therefore the next assertion is obvious.

Theorem 4.4.4. 1) *Let the function v be superharmonic (subharmonic). Then the function $(-v)$ is subharmonic (superharmonic).*
2) *Let the functions v_1 and v_2 be superharmonic (subharmonic). Then $v_1 + v_2$ is superharmonic (subharmonic).*

Theorem 4.4.5. *Let $u(x), v(x) \in C(\overline{D})$ and $u(x) \le v(x)$. Then $(u)_K \le (v)_K$ for each ball $K \subset D$.*

Proof. Denote $w = v - u$. The function $(w)_K$ is harmonic in K and $(w)_K \ge 0$ on the boundary ∂K. By the maximum principle, $(w)_K \ge 0$ in K, i.e. $(u)_K \le (v)_K$ in K. □

Theorem 4.4.6. *Let $v(x)$ be superharmonic in D. Then $v(x)$ attains its minimum on Σ. Moreover, if $v \not\equiv const$, then*

$$\min_{\xi\in\Sigma} v(\xi) < v(x)$$

for all $x \in D$.

Proof. Let $v \not\equiv const$, and let its minimum be attained at a point $x^0 \in D$, i.e.

$$v(x^0) = \min_{x\in\overline{D}} v(x) := m.$$

Let $K = K_R(x^0) \subset D$ be a ball around the point x^0 such that there exists $\tilde{x} \in S_R := \partial K_R$ for which $v(\tilde{x}) > v(x^0)$. Such a choice is possible since $v \not\equiv const.$. Denote $w = (v)_K$. Since $v(x)$ is superharmonic one has $w(x) \le v(x)$ and $w \not\equiv const.$. On the other hand, $v(x) \ge m$, and by Theorem 4.4.5, $w(x) \ge m$. Thus,

$$m \le w(x) \le v(x).$$

In particular, $w(x^0) = m$, i.e. the function $w(x)$ (which is harmonic in K) attains its minimum inside the ball. This contradiction proves the theorem. □

Theorem 4.4.7. *Let $v(x)$ be an upper function, and let $w(x)$ be a lower function. Then $v(x) \ge w(x)$ for all $x \in \overline{D}$.*

Proof. By virtue of Theorem 4.4.4, the function $v - w$ is superharmonic. Moreover, $(v-w)_{|\Sigma} \ge 0$. Then, by Theorem 4.4.6, $v - w \ge 0$ in $\overline{D}$. □

Theorem 4.4.8. *Let $v_1(x), \dots, v_n(x)$ be upper functions. Then*

$$v(x) := \min_{1\le l\le n} v_l(x)$$

is an upper function.

Proof. Let us show that $v(x)\in C(\overline{D})$. Take $x^0\in\overline{D}$. Since $v_l(x)\in C(\overline{D})$, then for each $\varepsilon>0$ there exists $\delta>0$ such that for $\|x-x^0\|\le\delta$ one has $v_l(x^0)-\varepsilon\le v_l(x)\le v_l(x^0)+\varepsilon$. Therefore,

$$v_l(x^0)-\varepsilon\le v_l(x)\ \to\ v(x^0)-\varepsilon\le v_l(x)\ \to\ v(x^0)-\varepsilon\le v(x),$$

$$v_l(x)\le v_l(x^0)+\varepsilon\ \to\ v(x)\le v_l(x^0)+\varepsilon\ \to\ v(x)\le v(x^0)+\varepsilon.$$

Thus, the function $v(x)$ is continuous at the point x^0. By virtue of the arbitrariness of x^0, we conclude that $v(x)\in C(\overline{D})$.

Furthermore, $(v_l)_K\le v_l$ for each ball $K\subset D$. Since $v\le v_l$, we have by Theorem 4.4.5: $(v)_K\le(v_l)_K$. Thus, $(v)_K\le v_l$, and consequently, $(v)_K\le v$, i.e. $v(x)$ is superharmonic in D. At last, since $v_{l|\Sigma}\ge\varphi$, we have $v_{|\Sigma}\ge\varphi$, i.e. $v(x)$ is an upper function. □

Theorem 4.4.9. *The cut-off function of an upper function is an upper function.*

Proof. Let $v(x)$ be an upper function, and let $K\subset D$ be a ball. Denote $z=(v)_K$. Clearly, $z\in C(\overline{D})$ and $z_{|\Sigma}\ge\varphi$. It remains to show that $(z)_{K_1}\le z$ for each ball $K_1\subset D$. Since $z=(v)_K\le v$, we have by Theorem 4.4.5, $(z)_{K_1}\le(v)_{K_1}\le v$. But $v(x)=z(x)$ for $x\notin K$. Therefore,

$$(z)_{K_1}(x)\le z(x)\quad\text{for}\quad x\notin K. \tag{4.4.4}$$

Let us now derive the same inequality for $x\in K$. For this purpose we consider 4 cases of the mutual location of the balls K and K_1.

1) Let $K\cap K_1=\emptyset$. If $x\in K$, then $x\notin K_1$, and consequently, $(z)_{K_1}(x)=z(x)$.

2) Let $K_1\subset K$. Since the function $z(x)$ is harmonic in K, it follows that

$$(z)_{K_1}(x)=z(x)$$

for $x\in K_1$. Outside K_1 we have: $(z)_{K_1}(x)\equiv z(x)$. Thus, $(z)_{K_1}(x)\equiv z(x)$ for all $x\in\overline{D}$.

3) Let $K\subset K_1$. The functions z and $(z)_{K_1}$ are harmonic in K and on the boundary (4.4.4) is valid. By the maximum principle, $(z)_{K_1}(x)\le z(x)$ for all $x\in K$.

4) Let the balls K and K_1 not contain each other and let $Q:=K\cap K_1\ne\emptyset$. The functions z and $(z)_{K_1}$ are harmonic in Q. Outside Q (and consequently, on the boundary ∂Q) the inequality $(z)_{K_1}(x)\le z(x)$ is obvious. By the maximum principle, we have $(z)_{K_1}(x)\le z(x)$ in Q. Theorem 4.4.9 is proved. □

We now consider the Dirichlet problem (4.4.1). Denote by E the set of upper functions. We note that $E\ne\emptyset$, since

$$v^*(x):=\max_{\xi\in\Sigma}\varphi(\xi)\in E.$$

Moreover, the set E is bounded from below, since if $v(x)\in E$, then

$$v(x)\ge\min_{\xi\in\Sigma}\varphi(\xi).$$

Consider the function

$$u(x)=\inf_{v\in E}v(x). \tag{4.4.5}$$

Clearly, the function $u(x)$ exists and is finite in $\overline{D}$.

Theorem 4.4.10 (The main theorem). *The function* $u(x)$, *defined by* (4.4.5), *is harmonic in* D.

Proof. It is sufficient to prove that u is harmonic in each ball $K \subset D$.

1) Let $K = K_R(x^0) \subset D$. Fix $\varepsilon > 0$. Let us construct a sequence of functions $\{v_n(x)\}_{n\ge 1}$ such that

$$\left.\begin{aligned} v_n \in E, \quad v_n(x) \text{ is harmonic in } K, \\ v_n(x) \le v_{n-1}(x) \text{ in } \overline{D}, \quad v_n(x^0) < u(x^0) + \frac{\varepsilon}{n}. \end{aligned}\right\} \tag{4.4.6}$$

For this purpose we choose $\tilde{v}_1 \in E$ such that $\tilde{v}_1(x^0) < u(x^0) + \varepsilon$, and put $v_1 = (\tilde{v}_1)_K$. By construction, v_1 is harmonic in K, $v_1 \le \tilde{v}_1$ in $\overline{D}$, and by Theorem 4.4.9, $v_1 \in E$. Clearly, $v_1(x^0) < u(x^0) + \varepsilon$.

Suppose now that the functions $v_1, \ldots, v_{n-1}$ with properties (4.4.6) are already constructed. We choose $\tilde{v}_n \in E$ such that $\tilde{v}_n(x^0) < u(x^0) + \varepsilon/n$, and put

$$v_n = \Big(\min(v_1, \ldots, v_{n-1}, \tilde{v}_n)\Big)_K.$$

By construction, v_n is harmonic in K, and by Theorems 4.4.8 and 4.4.9, $v_n \in E$. Furthermore, $v_n \le v_{n-1}$ and $v_n \le \tilde{v}_n$, and consequently, the function $v_n(x)$ satisfies (4.4.6).

In particular, it follows from (4.4.6) that

$$\lim_{n\to\infty} v_n(x^0) = u(x^0).$$

By Theorem 4.4.2, the sequence $\{v_n(x)\}$ converges in K:

$$\lim_{n\to\infty} v_n(x) = v(x), \quad x \in K$$

, $v(x)$ is harmonic in K, and the convergence is uniform in each closed domain $\overline{G} \subset K$. Since $v(x)$ is the limit of upper functions, we have $u(x) \le v(x)$ in $\overline{D}$. Moreover,

$$v(x^0) = u(x^0). \tag{4.4.7}$$

2) Let us show that $u(x) \equiv v(x)$, $x \in K$. Suppose on the contrary, that there exists $x^1 \in K$ such that $u(x^1) < v(x^1)$. Then there exists a function $z \in E$ such that

$$z(x^1) < v(x^1). \tag{4.4.8}$$

Denote

$$R_1 = \|x^0 - x^1\|, \quad K_1 = \{x : \|x - x^0\| < R_1\}.$$

Then $K_1 \subset K$ and $x^1 \in \partial K_1$. Consider the functions

$$\tilde{w} = \min(z, v), \quad w = (\tilde{w})_{K_1}.$$

Since $\tilde{w} \le v$, we infer from Theorem 4.4.5 that $(\tilde{w})_{K_1} \le (v)_{K_1} = v$, and consequently, $w(x) \le v(x)$, $x \in \overline{K_1}$. Moreover, by virtue of (4.4.8), $w(x^1) < v(x^1)$. By the maximum principle for harmonic functions, $w(x) < v(x)$ for all $x \in K_1$. In particular, this yields, in view of (4.4.7),

$$w(x^0) < u(x^0). \tag{4.4.9}$$

Consider now the functions

$$w_n = \Big(\min(z, v_n)\Big)_{K_1}.$$

By virtue of Theorems 4.4.8-4.4.9, $w_n \in E$. Let us show that

$$\lim_{n\to\infty} w_n(x) = w(x) \quad \text{uniformly in } \overline{K_1}. \tag{4.4.10}$$

Indeed, since $\{v_n(x)\}$ converges to $v(x)$ uniformly in $\overline{K_1}$, we have that for each $\varepsilon > 0$ there exists N such that

$$v(x) - \varepsilon \le v_n(x) \le v(x) + \varepsilon$$

for all $n > N$, $x \in \overline{K_1}$. Therefore,

$$\min(z, v - \varepsilon) \le \min(z, v_n) \le \min(z, v + \varepsilon)$$

or

$$\min(z, v) - \varepsilon \le \min(z, v_n) \le \min(z, v) + \varepsilon.$$

By virtue of Theorem 4.4.5,

$$\Big(\min(z, v)\Big)_{K_1} - \varepsilon \le \Big(\min(z, v_n)\Big)_{K_1} \le \Big(\min(z, v)\Big)_{K_1} + \varepsilon$$

or

$$w(x) - \varepsilon \le w_n(x) \le w(x) + \varepsilon,$$

i.e. (4.4.10) is proved. It follows from (4.4.9)-(4.4.10) that $w_n(x^0) < u(x^0)$ for sufficiently large n, which is impossible in view of (4.4.5). Thus, $u(x) \equiv v(x)$, $x \in K$, and consequently, $u(x)$ is harmonic in K. □

Theorem 4.4.11. *Let the function $u(x)$ be defined by $(4.4.5)$. The Dirichlet problem $(4.4.1)$ has a solution if and only if $u(x) \in C(\overline{D})$ and $u_{|\Sigma} = \varphi$.*

Proof. Clearly, if $u(x) \in C(\overline{D})$ and $u_{|\Sigma} = \varphi$, then $u(x)$ is the solution of the problem (4.4.1). Conversely, suppose that the solution of problem (4.4.1) exists; denote it by $\tilde{u}(x)$. Then $\tilde{u}(x)$ is harmonic in D, $\tilde{u}(x) \in C(\overline{D})$ and $\tilde{u}_{|\Sigma} = \varphi$, hence the function $\tilde{u}(x)$ is upper and lower simultaneously. Since $\tilde{u}(x)$ is an lower function, we have $\tilde{u}(x) \le u(x)$. Since $\tilde{u}(x)$ is an upper function, we have $\tilde{u}(x) \ge u(x)$. Therefore, $\tilde{u}(x) = u(x)$, and consequently, $u(x) \in C(\overline{D})$, $u_{|\Sigma} = \varphi$. □

Thus, it remains to find out when the function $u(x)$, defined by (4.4.5), is continuous in $\overline{D}$ and $u_{|\Sigma} = \varphi$. The answer on this question depends on the properties of $\varphi(x)$ and of Σ. For example, if $\varphi(x) \equiv 1$, then for each Σ the problem (4.4.1) has the solution $u(x) \equiv 1$. However, we are interested in another question:
When does the solution of problem (4.4.1) exist for each $\varphi(x) \in C(\Sigma)$?
In this case the answer will depend only on the properties of Σ, i.e. on the configuration of the domain.

Definition 4.4.4. A point $x^0 \in \Sigma$ is called *regular*, if for each $\varphi \in C(\Sigma)$:

$$\lim_{x \to x^0, x \in D} u(x) = \varphi(x^0),$$

where the function $u(x)$ is defined by (4.4.5).

The next theorem is obvious.

Theorem 4.4.12. *The Dirichlet problem* (4.4.1) *has a solution for each* $\varphi(x) \in C(\Sigma)$ *if and only if all points of* Σ *are regular.*

Remark 4.4.1. If on Σ there are non-regular points, then for some φ the Dirichlet problem has no solutions, and for some φ (for example, for $\varphi(x) \equiv 1$) the Dirichlet problem has a solution. We provide sufficient conditions for the regularity of a point.

Definition 4.4.5. Let $x^0 \in \Sigma$. The function $\omega(x)$ is called *barrier* for x^0, if $\omega(x)$ is superharmonic in $\overline{D}$, $\omega(x^0) = 0$ and $\omega(x) > 0$ for $x \in \overline{D}, x \neq x^0$.

Theorem 4.4.13. *If for a point* $x^0 \in \Sigma$ *there exists a barrier, then* x^0 *is regular.*

Proof. For $\delta > 0$ we denote $Q_\delta = \overline{D} \cap \overline{K_\delta(x^0)}$, $\Sigma_\delta = Q_\delta \cap \Sigma$. Fix $\varepsilon > 0$. Since $\varphi(x) \in C(\Sigma)$, there exists $\delta > 0$ such that for $x \in \Sigma_\delta$ we have

$$\varphi(x^0) - \varepsilon \leq \varphi(x) \leq \varphi(x^0) + \varepsilon. \tag{4.4.11}$$

Moreover,

$$\omega(x) \geq h > 0 \quad \text{for} \quad x \in \overline{D \setminus Q_\delta}. \tag{4.4.12}$$

Consider the functions

$$f(x) = \varphi(x^0) + \varepsilon + C\omega(x)$$

and

$$g(x) = \varphi(x^0) - \varepsilon - C\omega(x),$$

with $C > 0$. Let us show that one can choose $C > 0$ such that $f(x)$ is an upper function, and $g(x)$ is a lower one. For definiteness, we confine ourselves only to the consideration of $f(x)$, since for $g(x)$ arguments are similar. Clearly, $f(x)$ is superharmonic in $\overline{D}$. It follows from (4.4.11) that $f(x) \geq \varphi(x)$, $x \in \Sigma_\delta$ for all $C > 0$. For $x \in \Sigma \setminus \Sigma_\delta$, by virtue of (4.4.12), we have $f(x) \geq \varphi(x^0) + \varepsilon + C$. Choose $C > 0$ such that

$$f(x) \geq \min_{\xi \in \Sigma} \varphi(\xi).$$

Then $f(x)$ is an upper function.

Since $f(x)$ is upper, and $g(x)$ is lower, we have

$$\varphi(x^0) - \varepsilon - C\omega(x) \leq u(x) \leq \varphi(x^0) + \varepsilon + C\omega(x) \quad \text{for all} \quad x \in \overline{D}.$$

As $x \to x^0$ this yields

$$\varphi(x^0) - \varepsilon \leq \underline{\lim}_{x \to x^0} u(x) \leq \overline{\lim}_{x \to x^0} u(x) \leq \varphi(x^0) + \varepsilon.$$

As $\varepsilon \to 0$ we deduce that there exists $\lim\limits_{x\to x^0} u(x)$, and

$$\lim_{x\to x^0} u(x) = \varphi(x^0).$$

Theorem 4.4.13 is proved. □

Theorem 4.4.14. *Let $x^0 \in \Sigma$, and let there exist a closed ball $\overline{K_R(x^1)}$ with its center outside $\overline{D}$ such that $\overline{K_R(x^1)} \cap \overline{D} = \{x^0\}$ (in this case we shall say that the point x^0 can be "reached" by a ball). Then for the point x^0 there exist a barrier.*

Proof. We consider the function

$$\omega(x) = \frac{1}{R} - \frac{1}{r},$$

where $r = \|x - x^1\|$. Clearly, $\omega(x) \in C(\overline{D})$ and $\omega(x)$ is harmonic in D, hence $\omega(x)$ is superharmonic. Furthermore, $\omega(x^0) = 0$ and $\omega(x) > 0$ for $x \in \overline{D}, x \neq x^0$. Thus, the function $\omega(x)$ is a barrier for the point x^0. □

Conclusion. Assume that each point of Σ can be "reached" by a ball. Then for each $\varphi(x) \in C(\Sigma)$ the solution of the Dirichlet problem (4.4.1) exists, is unique and it is given by (4.4.5).

Thus, we have established the solvability of the Dirichlet problem (4.4.1) for a wide class of domains (for example, for all convex domains). We note that one can obtain weaker sufficient conditions of regularity of a point. For example, if the point $x^0 \in \Sigma$ can be "reached" by a non-degenerate cyclic cone, then the point x^0 is regular [4, Chapter 3].

4.5. The Dirichlet Problem for the Poisson Equation

Let $D \subset \mathbf{R}^3$ be a bounded domain with the boundary $\Sigma \in PC^1$. We consider the following problem

$$\Delta u = f(x) \quad (x \in D), \tag{4.5.1}$$

$$u_{|\Sigma} = \varphi(x). \tag{4.5.2}$$

Suppose that we know a particular solution $v(x)$ of equation (4.5.1). Then the solution of problem (4.5.1)-(4.5.2) has the form $u(x) = v(x) + w(x)$, where $w(x)$ is the solution of the Dirichlet problem

$$\left.\begin{aligned} &\Delta w = 0 \quad (x \in D), \\ &w_{|\Sigma} = \varphi_1, \end{aligned}\right\} \tag{4.5.3}$$

where

$$\varphi_1 := \varphi - v_{|\Sigma}.$$

The Dirichlet problem (4.5.3) for the Laplace equation was studied in Sections 4.3-4.4. Thus, in order to solve the Dirichlet problem (4.5.1)-(4.5.2) for the Poisson equation it is sufficient to find a particular solution of equation (4.5.1). We consider the function

$$u(x) = -\frac{1}{4\pi}\int_D \frac{f(\xi)}{r}\,d\xi, \quad r = \|x - \xi\|. \tag{4.5.4}$$

Let us show that $\Delta u = f$, i.e (4.5.4) gives us the desired particular solution of (4.5.1). The integral in (4.5.4) is called the *volume potential*, since it describes the potential of gravitational field created by the body D with the density $f(\xi)$. Denote as usual by $K_\delta(x)$ the ball of radius δ around the point x, $S_\delta(x) = \partial K_\delta(x)$ is the sphere. We have

$$\left.\begin{aligned} \int_{K_\delta(x)} \frac{d\xi}{r} = \int_0^\delta \frac{dr}{r} \int_{S_r(x)} ds = \int_0^\delta 4\pi r\, dr = 2\pi\delta^2, \\ \int_{K_\delta(x)} \frac{d\xi}{r^2} = \int_0^\delta \frac{dr}{r^2} \int_{S_r(x)} ds = 4\pi\delta. \end{aligned}\right\} \tag{4.5.5}$$

Theorem 4.5.1. *Let* $f(x) \in C^1(\overline{D})$. *Then the function* $u(x)$, *defined by* (4.5.4), *is a particular solution of the Poisson equation* (4.5.1).

Proof. 1) Since $|f(\xi)| \le M$ in $\overline{D}$, it follows from (4.5.4)-(4.5.5) that for any $x \in \overline{D}$,

$$|u(x)| \le \frac{M}{4\pi} \int_{K_\delta(x)} \frac{d\xi}{r} + \frac{M}{4\pi\delta} \int_{D\setminus K_\delta(x)} d\xi \le \frac{M\delta^2}{2} + \frac{MV}{4\pi\delta},$$

where V is the volume of D. Therefore, the integral in (4.5.4) converges absolutely and uniformly in $\overline{D}$, and the function $u(x)$ exists and is bounded in $\overline{D}$.

Let us show that $u(x) \in C(\overline{D})$. Let $y \in K_{\delta/2}(x)$. By virtue of (4.5.4),

$$|u(y) - u(x)| \le \frac{M}{4\pi} \int_{K_{\delta/2}(x)} \left| \frac{1}{r_1} - \frac{1}{r} \right| d\xi + \frac{M}{4\pi} \int_{D\setminus K_{\delta/2}(x)} \left| \frac{1}{r_1} - \frac{1}{r} \right| d\xi$$

$$=: I_\delta + J_\delta,$$

where $r = \|x - \xi\|, r_1 = \|y - \xi\|$. Using (4.5.5) we calculate

$$I_\delta \le \frac{M}{4\pi} \int_{K_{\delta/2}(x)} \frac{d\xi}{r_1} + \frac{M}{4\pi} \int_{K_{\delta/2}(x)} \frac{d\xi}{r}$$

$$\le \frac{M}{4\pi} \int_{K_\delta(y)} \frac{d\xi}{r_1} + \frac{M}{4\pi} \int_{K_{\delta/2}(x)} \frac{d\xi}{r} \le \frac{3M\delta^2}{4}.$$

Take $\varepsilon > 0$. Let $\delta = \sqrt{2\varepsilon/(3M)}$. Then $I_\delta \le \varepsilon/2$. In the domain $D \setminus K_\delta(x)$ the function $1/r$ is continuous, and consequently, there exists δ_1 $(\delta_1 < \delta/2)$ such that

$$\left| \frac{1}{r_1} - \frac{1}{r} \right| \le \frac{2\pi\varepsilon}{MV}$$

for all $y \in K_{\delta_1}(x)$; hence $J_\delta \le \varepsilon/2$. Thus,

$$\forall \varepsilon > 0 \quad \exists \delta_1 > 0 \qquad y \in K_{\delta_1}(x) \to |u(y) - u(x)| \le \varepsilon.$$

By virtue of the arbitrariness of $x \in \overline{D}$ we conclude that $u(x) \in C(\overline{D})$.

2) Denote

$$v_k(x) := \frac{\partial u}{\partial x_k}, \quad k = \overline{1,3}.$$

Differentiating (4.5.4) formally, we obtain

$$v_k(x) = -\frac{1}{4\pi}\int_D f(\xi)\frac{\partial}{\partial x_k}\left(\frac{1}{r}\right)d\xi, \quad x\in\overline{D}, \quad r=\|x-\xi\|. \tag{4.5.6}$$

Since

$$\frac{\partial}{\partial x_k}\left(\frac{1}{r}\right) = \frac{\xi_k - x_k}{r^3}, \quad \frac{|\xi_k - x_k|}{r}\le 1, \quad |f(\xi)|\le M,$$

it follows from (4.5.6) and (4.5.5) that

$$|v_k(x)| \le \frac{M}{4\pi}\int_{K_\delta(x)}\frac{d\xi}{r^2} + \frac{M}{4\pi\delta^2}\int_{D\setminus K_\delta(x)} d\xi \le M\delta + \frac{MV}{4\pi\delta^2}.$$

Therefore, the integral in (4.5.6) converges absolutely and uniformly in $\overline{D}$, and the functions $v_k(x) = \frac{\partial u}{\partial x_k}$ exist and are finite in $\overline{D}$.

Let us show that $v_k(x)\in C(\overline{D})$, $k=\overline{1,3}$. Let $y\in K_{\delta/2}(x)$. By virtue of (4.5.6),

$$|v_k(y)-v_k(x)| \le \frac{M}{4\pi}\int_{K_{\delta/2}(x)}\left|\frac{\xi_k-y_k}{r_1^3}-\frac{\xi_k-x_k}{r^3}\right|d\xi$$

$$+\frac{M}{4\pi}\int_{D\setminus K_{\delta/2}(x)}\left|\frac{\xi_k-y_k}{r_1^3}-\frac{\xi_k-x_k}{r^3}\right|d\xi$$

$$= I_\delta^1 + J_\delta^1,$$

where $r=\|x-\xi\|, r_1=\|y-\xi\|$. Using (4.5.5) we calculate

$$I_\delta^1 \le \frac{M}{4\pi}\int_{K_{\delta/2}(x)}\frac{d\xi}{r_1^2} + \frac{M}{4\pi}\int_{K_{\delta/2}(x)}\frac{d\xi}{r^2}$$

$$\le \frac{M}{4\pi}\int_{K_\delta(y)}\frac{d\xi}{r_1^2} + \frac{M}{4\pi}\int_{K_{\delta/2}(x)}\frac{d\xi}{r^2} \le \frac{3M\delta}{2}.$$

Take $\varepsilon>0$. Let $\delta=\varepsilon/(3M)$. Then $I_\delta^1\le\varepsilon/2$. In the domain $D\setminus K_{\delta/2}(x)$ the function $\frac{\partial}{\partial x_k}(\frac{1}{r})$ is continuous, and consequently, there exists δ_1 $(\delta_1<\delta/2)$ such that $J_\delta^1\le\varepsilon/2$ for $y\in K_{\delta_1}(x)$. Thus,

$$\forall\varepsilon>0 \quad \exists\delta_1>0 \qquad y\in K_{\delta_1}(x)\to|v_k(y)-v_k(x)|\le\varepsilon.$$

Since $x\in\overline{D}$ is arbitrary, we conclude that $v_k(x)\in C(\overline{D})$, i.e. $u(x)\in C^1(\overline{D})$.

3) Denote

$$w_k(x) := \frac{\partial^2 u}{\partial x_k^2}, \quad k=\overline{1,3}.$$

Differentiating (4.5.4) formally, we obtain

$$w_k(x) = -\frac{1}{4\pi}\int_D f(\xi)\frac{\partial^2}{\partial x_k^2}\left(\frac{1}{r}\right)d\xi, \quad x\in\overline{D}. \tag{4.5.7}$$

Denote

$$f_k(\xi) = \frac{\partial f(\xi)}{\partial \xi_k}.$$

Since

$$\frac{\partial}{\partial \xi_k}\left(\frac{1}{r}\right) = -\frac{\partial}{\partial x_k}\left(\frac{1}{r}\right),$$

then using the Gauß-Ostrogradskii formula we calculate

$$\begin{aligned}
Q_{\delta,k} &:= -\frac{1}{4\pi}\int_{K_\delta(x)} f(\xi)\frac{\partial^2}{\partial x_k^2}\left(\frac{1}{r}\right) d\xi = \frac{1}{4\pi}\int_{K_\delta(x)} f(\xi)\frac{\partial}{\partial \xi_k}\left(\frac{\partial}{\partial x_k}\left(\frac{1}{r}\right)\right) d\xi \\
&= \frac{1}{4\pi}\int_{K_\delta(x)} \frac{\partial}{\partial \xi_k}\left(f(\xi)\frac{\partial}{\partial x_k}\left(\frac{1}{r}\right)\right) d\xi - \frac{1}{4\pi}\int_{K_\delta(x)} f_k(\xi)\frac{\partial}{\partial x_k}\left(\frac{1}{r}\right) d\xi \\
&= \frac{1}{4\pi}\int_{S_\delta(x)} f(\xi)\frac{\partial}{\partial x_k}\left(\frac{1}{r}\right)\cos(n,\xi_k)\, ds - \frac{1}{4\pi}\int_{K_\delta(x)} f_k(\xi)\frac{\partial}{\partial x_k}\left(\frac{1}{r}\right) d\xi.
\end{aligned}$$

Since

$$\frac{\partial}{\partial x_k}\left(\frac{1}{r}\right) = \frac{\xi_k - x_k}{r^3}, \quad \frac{\xi_k - x_k}{r} = \cos(n,\xi_k),$$

we have

$$\begin{aligned}
Q_{\delta,k} := \frac{1}{4\pi\delta^2}\int_{S_\delta(x)} & f(\xi)\cos^2(n,\xi_k)\, ds \\
-\frac{1}{4\pi}\int_{K_\delta(x)} & f_k(\xi)\frac{\xi_k - x_k}{r^3}\, d\xi. \qquad (4.5.8)
\end{aligned}$$

There exists a constant $M > 0$ such that $|f(\xi)| \le M$, $|f_k(\xi)| \le M$. Then, in view of (4.5.5), we have

$$|Q_{\delta,k}| \le \frac{M}{4\pi\delta^2}\int_{S_\delta(x)} ds + \frac{M}{4\pi}\int_{K_\delta(x)} \frac{d\xi}{r^2} \le (1+\delta)M,$$

and consequently,

$$|w_k(x)| \le (1+\delta)M + \frac{M}{4\pi}\int_{D\setminus K_\delta(x)} \left|\frac{\partial^2}{\partial x_k^2}\left(\frac{1}{r}\right)\right| d\xi \le C_\delta.$$

Thus, the integral in (4.5.7) converges absolutely and uniformly in $\overline{D}$, and the functions $w_k(x) := \dfrac{\partial^2 u}{\partial x_k^2}$ exist and are bounded in $\overline{D}$. Using (4.5.7) and (4.5.8) we calculate

$$\begin{aligned}
\Delta u(x) := \sum_{k=1}^{3}\frac{\partial^2 u}{\partial x_k^2} &= \frac{1}{4\pi\delta^2}\int_{S_\delta(x)} f(\xi)\, ds - \frac{1}{4\pi}\sum_{k=1}^{3}\int_{K_\delta(x)} f_k(\xi)\frac{\xi_k - x_k}{r^3}\, d\xi \\
&= f(x) + \Omega_{\delta,1} + \Omega_{\delta,2},
\end{aligned}$$

where

$$\Omega_{\delta,1} := \frac{1}{4\pi\delta^2}\int_{S_\delta(x)} (f(\xi) - f(x))\, ds,$$

$$\Omega_{\delta,2} := -\frac{1}{4\pi}\sum_{k=1}^{3}\int_{K_\delta(x)} f_k(\xi)\frac{\xi_k - x_k}{r^3}\,d\xi.$$

By virtue of (4.5.5),

$$|\Omega_{\delta,2}| \le \frac{3M}{4\pi}\int_{K_\delta(x)}\frac{d\xi}{r^2} \le 3M\delta.$$

Take $\varepsilon > 0$. Since $f(\xi) \in C(\overline{D})$, there exists $\delta > 0$ such that $|f(\xi) - f(x)| \le \varepsilon$ for $\xi \in S_\delta(x)$. Then $|\Omega_{\delta,1}| \le \varepsilon$. Thus, $|\Delta u(x) - f(x)| \le 3M\delta + \varepsilon$, and consequently, $\Delta u = f$. Theorem 4.5.1 is proved. □

4.6. The Method of Integral Equations

1. Let for definiteness $n = 3$, and let $D \subset \mathbf{R}^3$ be a bounded domain with the boundary $\Sigma \in PC^1$, $D_1 = \mathbf{R}^3 \setminus \overline{D}$. We consider the functions

$$Q(x) = \int_\Sigma \frac{q(\xi)}{r}\,ds, \quad F(x) = \int_\Sigma f(\xi)\frac{\partial_\xi(\frac{1}{r})}{\partial n_\xi}\,ds, \tag{4.6.1}$$

where $q(\xi), f(\xi) \in C(\Sigma)$, $r = \|\xi - x\|$ and n_ξ is the outer normal to Σ at the point ξ. The function $Q(x)$ is called the *single-layer potential* with the density $q(\xi)$. The function $F(x)$ is called the *double-layer potential* with the density $f(\xi)$.

Physical sense: $Q(x)$ is the potential of the field created by charges distributed on Σ with the density $q(\xi)$, and $F(x)$ is the potential of the field created by the dipole distribution on Σ with the density $f(\xi)$.

Idea of the method: The basic integral formula for harmonic functions (see Section 4.1) contains terms of the form (4.6.1). We will seek solutions of the Dirichlet and Neumann problems in the form (4.6.1). As a result we will obtain some integral equations for finding q and f. Beforehand we will study properties of the functions defined by (4.6.1).

2. Auxiliary assertions

Everywhere in Section 4.6 we will assume that $\Sigma \in C^2$, i.e. Σ has a continuous and bounded curvature. This means that for each fixed $x \in \Sigma$ one can choose a local coordinate system $(\alpha_1, \alpha_2, \alpha_3)$ such that the origin coincides with the point x, the plane (α_1, α_2) coincides with the tangent plane to Σ at the point x, and the axis α_3 coincides with the outer normal $\bar{n}_x$ to Σ at the point x (see fig. 4.6.1). Moreover, there exists $\delta_0 > 0$ (independent of x) such that the part of the surface $\Sigma_{\delta_0}(x) := \Sigma \cap K_{\delta_0}(x)$ is represented by a single-valued function $\alpha_3 = \Phi(\alpha_1, \alpha_2)$, $\Phi \in C^2$ and $\left|\dfrac{\partial^2\Phi}{\partial\alpha_k\partial\alpha_j}\right| \le \Phi_0$ for $\rho := \sqrt{\alpha_1^2 + \alpha_2^2} \le \delta_0$, where Φ_0 does not depend on x (it is the maximal curvature). In the sequel, we assume that $\delta_0 < 1/(8\Phi_0)$. Since $\Phi(0,0) = \Phi_{\alpha_1}(0,0) = \Phi_{\alpha_2}(0,0) = 0$, we have by virtue of the Taylor formula:

$$|\Phi(\alpha_1, \alpha_2)| \le \Phi_0\rho^2, \ |\Phi_{\alpha_k}(\alpha_1, \alpha_2)| \le 2\Phi_0\rho \text{ for } \rho := \sqrt{\alpha_1^2 + \alpha_2^2} \le \delta_0. \tag{4.6.2}$$

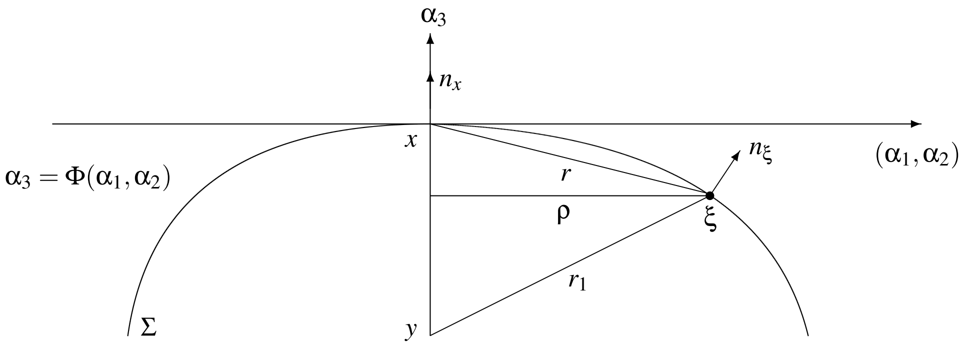

Figure 4.6.1.

Lemma 4.6.1. *Let* $x \in \Sigma$, $\xi \in \Sigma_{\delta_0}$, $r = \|x - \xi\|$. *Then*

$$|\sin(n_x, n_\xi)| \leq 4\Phi_0 r, \quad |\cos(n_x, n_\xi)| \geq 1/2. \tag{4.6.3}$$

Proof. Let $\bar{r} = (\alpha_1, \alpha_2, \Phi(\alpha_1, \alpha_2))$ be the local coordinates of the point ξ. Then, using (4.6.2), we calculate

$$|\sin(n_x, n_\xi)| \leq |\mathrm{tg}(n_x, n_\xi)| = \left|\frac{\partial \Phi(\alpha_1, \alpha_2)}{\partial \bar{r}}\right|$$

$$|\Phi_{\alpha_1}(\alpha_1, \alpha_2)\cos(\alpha_1, \bar{r}) + \Phi_{\alpha_2}(\alpha_1, \alpha_2)\cos(\alpha_2, \bar{r})|$$
$$\leq |\Phi_{\alpha_1}(\alpha_1, \alpha_2)| + |\Phi_{\alpha_2}(\alpha_1, \alpha_2)| \leq 4\Phi_0 \rho.$$

Since

$$r = \sqrt{\alpha_1^2 + \alpha_2^2 + (\Phi(\alpha_1, \alpha_2))^2} \geq \sqrt{\alpha_1^2 + \alpha_2^2} := \rho,$$

we obtain

$$|\sin(n_x, n_\xi)| \leq 4\Phi_0 r.$$

From $4\Phi_0 r \leq 4\Phi_0 \delta_0 \leq 1/2$, we infer $|\cos(n_x, n_\xi)| \geq 1/2$. □

Lemma 4.6.2. *Let* $x \in \Sigma$, $\xi \in \Sigma_{\delta_0}$, $r = \|x - \xi\|$. *Then*

$$\left|\frac{\partial_x r}{\partial n_x}\right| \leq \Phi_0 r, \quad \left|\frac{\partial_\xi r}{\partial n_\xi}\right| \leq \Phi_0 r, \tag{4.6.4}$$

$$\left|\frac{\partial_x(\frac{1}{r})}{\partial n_x}\right| \leq \frac{\Phi_0}{r}, \quad \left|\frac{\partial_\xi(\frac{1}{r})}{\partial n_\xi}\right| \leq \frac{\Phi_0}{r}. \tag{4.6.5}$$

Proof. By Lemma 4.3.1,

$$\frac{\partial_\xi r}{\partial n_\xi} = \cos(n_\xi, \bar{r}), \quad \frac{\partial_x r}{\partial n_x} = -\cos(n_x, \bar{r}),$$

where

$$\bar{r} = (\xi_1 - x_1, \xi_2 - x_2, \xi_3 - x_3)$$

. Let $(\alpha_1,\alpha_2,\Phi(\alpha_1,\alpha_2))$ be the local coordinates of the point ξ. Then

$$|\cos(n_x,\bar r)| = \frac{|\Phi(\alpha_1,\alpha_2)|}{r}$$

(see fig. 4.6.1). Using (4.6.2) and the inequality $\rho \le r$, we obtain

$$\left|\frac{\partial_x r}{\partial n_x}\right| = |\cos(n_x,\bar r)| = \frac{|\Phi(\alpha_1,\alpha_2)|}{r} \le \frac{\Phi_0\rho^2}{r} \le \Phi_0 r.$$

By symmetry, the second formula in (4.6.4) follows from the first one. Formulae (4.6.5) are evident corollaries of (4.6.4). □

Lemma 4.6.3. *Let* $x\in\Sigma$, $\delta\le\delta_0$. *Then*

$$\int_{\Sigma_\delta(x)} \frac{ds}{r} \le 4\pi\delta,\ r:=\|x-\xi\|,\ \Sigma_\delta(x):=\Sigma\cap K_\delta(x),\ \xi\in\Sigma_\delta(x). \tag{4.6.6}$$

Moreover, if $y\in K_{\delta/2}(x)$, $\xi\in\Sigma_{\delta/2}(x)$, $r_1:=\|y-\xi\|$, *then*

$$\int_{\Sigma_{\delta/2}(x)} \frac{ds}{r_1} \le 4\pi\delta.$$

Proof. Let $(\alpha_1,\alpha_2,\Phi(\alpha_1,\alpha_2))$ be the local coordinates of the point ξ and $\rho := \sqrt{\alpha_1^2+\alpha_2^2}$. Denote by $\sigma_\delta=\{(\alpha_1,\alpha_2):\ \rho\le\delta\}$ the disc of radius δ. Using (4.6.3) and the inequality $\rho\le r$, we obtain

$$\int_{\Sigma_\delta(x)} \frac{ds}{r} \le \int_{\Sigma_\delta(x)} \frac{ds}{\rho} = \int_{\sigma_\delta} \frac{d\alpha_1 d\alpha_2}{\rho\cos(n_x,n_\xi)}$$

$$\le 2\int_{\sigma_\delta} \frac{d\alpha_1 d\alpha_2}{\rho} = 2\int_0^\delta \frac{d\rho}{\rho}\,2\pi\rho = 4\pi\delta.$$

Furthermore, let $(\beta_1,\beta_2,\beta_3)$ be the local coordinates of the point y. Then

$$r_1 = \sqrt{(\alpha_1-\beta_1)^2+(\alpha_2-\beta_2)^2+(\Phi(\alpha_1,\alpha_2)-\beta_3)^2}$$

$$\ge \sqrt{(\alpha_1-\beta_1)^2+(\alpha_2-\beta_2)^2} := \tilde\rho,$$

and consequently,

$$\int_{\Sigma_{\delta/2}(x)} \frac{ds}{r_1} \le \int_{\Sigma_{\delta/2}(x)} \frac{ds}{\tilde\rho} \le 2\int_{\sigma_{\delta/2}} \frac{d\alpha_1 d\alpha_2}{\tilde\rho}$$

$$\le 2\int_{\sigma_\delta(\beta)} \frac{d\alpha_1 d\alpha_2}{\tilde\rho} \le 2\int_{\sigma_\delta} \frac{d\alpha_1 d\alpha_2}{\rho} = 4\pi\delta,$$

where $\sigma_\delta(\beta)=\{(\alpha_1,\alpha_2):\ \tilde\rho\le\delta)\}$ is the disc of radius δ around the point (β_1,β_2). Lemma 4.6.3 is proved. □

Lemma 4.6.4. *Let* $x \in \Sigma,\ \delta \le \delta_0,\ \xi \in \Sigma_\delta(x)$. *Then*

$$\int_{\Sigma_\delta(x)} \left| \frac{\partial_\xi(\frac{1}{r})}{\partial n_\xi} \right| ds \le 4\pi\Phi_0\delta, \quad \int_{\Sigma_\delta(x)} \left| \frac{\partial_x(\frac{1}{r})}{\partial n_x} \right| ds \le 4\pi\Phi_0\delta,\ r := \|x-\xi\|. \tag{4.6.7}$$

Formulae (4.6.7) are obvious consequences of the formulae (4.6.5) and (4.6.6).

Lemma 4.6.5. *Let* $x \in \Sigma, \delta \le \delta_0, y \in K_\delta(x), y = x + \alpha n_x, \alpha \in \mathbf{R}, r_1 = \|y-\xi\|$. *Then*

$$\int_{\Sigma_\delta(x)} \left| \frac{\partial_\xi(\frac{1}{r_1})}{\partial n_\xi} + \frac{\partial_y(\frac{1}{r_1})}{\partial n_x} \right| ds \le 64\pi\Phi_0\delta. \tag{4.6.8}$$

Proof. Let $\xi \in \Sigma_\delta(x)$. Then, by Lemma 4.3.1,

$$\frac{\partial_\xi(r_1)}{\partial n_\xi} + \frac{\partial y(r_1)}{\partial n_x} = \cos(n_\xi, \bar{r}_1) - \cos(n_x, \bar{r}_1),$$

where

$$\bar{r}_1 = (\xi_1 - y_1, \xi_2 - y_2, \xi_3 - y_3).$$

Together with Lemma 4.6.1 this yields

$$\left| \frac{\partial_\xi(r_1)}{\partial n_\xi} + \frac{\partial y(r_1)}{\partial n_x} \right| \le 2|\sin(n_x, n_\xi)| \le 8\Phi_0 r, \text{ where } r = \|x-\xi\|. \tag{4.6.9}$$

Let $(\alpha_1, \alpha_2, \Phi(\alpha_1, \alpha_2))$ be the local coordinates of the point ξ. Since $y = x + \alpha n_x$, we have $\rho \le r_1$ (see fig. 4.6.1). Using (4.6.2) we calculate

$$r = \sqrt{\alpha_1^2 + \alpha_2^2 + (\Phi(\alpha_1, \alpha_2))^2} \le \sqrt{\rho^2 + \Phi_0^2\rho^4} \le \rho\sqrt{1 + \Phi_0^2\delta_0^2} \le 2\rho \le 2r_1.$$

Together with (4.6.9) this yields

$$\left| \frac{\partial_\xi(r_1)}{\partial n_\xi} + \frac{\partial y(r_1)}{\partial n_x} \right| \le 16\Phi_0 r_1.$$

Therefore

$$\int_{\Sigma_\delta(x)} \left| \frac{\partial_\xi(\frac{1}{r_1})}{\partial n_\xi} + \frac{\partial_y(\frac{1}{r_1})}{\partial n_x} \right| ds \le 16\Phi_0 \int_{\Sigma_\delta(x)} \frac{ds}{r_1}$$

$$\le 16\Phi_0 \int_{\Sigma_\delta(x)} \frac{ds}{\rho} = 16\Phi_0 \int_{\sigma_\delta} \frac{d\alpha_1 d\alpha_2}{\rho\cos(n_x, n_\xi)}$$

$$\le 32\Phi_0 \int_{\sigma_\delta} \frac{d\alpha_1 d\alpha_2}{\rho} = 64\pi\Phi_0\delta,$$

i.e. (4.6.8) is valid. Lemma 4.6.5 is proved. □

Lemma 4.6.6. *Let* $x \in \Sigma, \delta \le \delta_0, y \in K_\delta(x), y \notin \Sigma$. *Then*

$$\int_{\Sigma_\delta(x)} \left| \frac{\partial_\xi(\frac{1}{r_1})}{\partial n_\xi} \right| ds \le C_*,\ r_1 := \|y-\xi\|,\ \xi \in \Sigma_\delta(x), \tag{4.6.10}$$

where C_* *depends on* Σ *and does not depend on* x, y, δ.

Proof. Let Ω be a simply connected part of Σ_δ, on which the function $\dfrac{\partial_\xi(\frac{1}{r_1})}{\partial n_\xi}$ preserves the sign. Let B be the cone with vertex at the point y and with the basis Ω. Fix $\varepsilon > 0$ and consider the domain $B_\varepsilon = \{\xi \in B : \ r_1 \ge \varepsilon\}$ (see fig. 4.6.2).

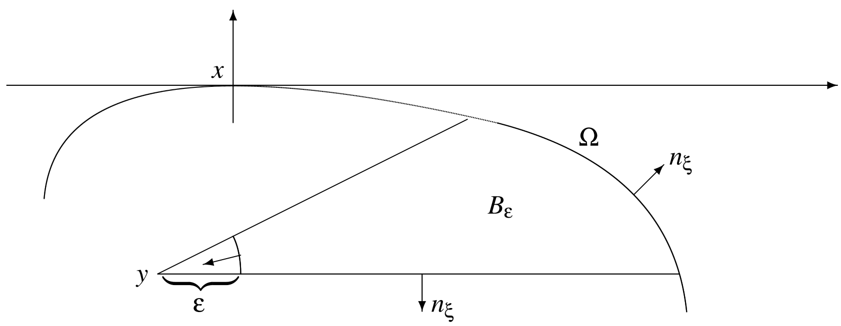

Figure 4.6.2.

Clearly, $\partial B_\varepsilon = \Omega \cup \Omega_0 \cup S^0_\varepsilon(y)$, where Ω_0 is the lateral surface of B_ε, $S^0_\varepsilon(y) = S_\varepsilon(y) \cap B$ is the part of the sphere $r_1 = \varepsilon$, lying in B. In B_ε the function $1/r_1$ is harmonic with respect to ξ. Hence, by Theorem 4.1.2,

$$\left(\int_\Omega + \int_{\Omega_0} + \int_{S^0_\varepsilon(y)}\right) \frac{\partial_\xi(\frac{1}{r_1})}{\partial n_\xi}\, ds = 0.$$

Since $n_\xi \perp \bar{r}_1$ on Ω_0, we have

$$\int_{\Omega_0} \frac{\partial_\xi(\frac{1}{r_1})}{\partial n_\xi}\, ds = -\int_{\Omega_0} \frac{\cos(n_\xi, \bar{r}_1)}{r_1^2}\, ds = 0.$$

On $S^0_\varepsilon(y)$ we have

$$\frac{\partial_\xi(\frac{1}{r_1})}{\partial n_\xi} = -\frac{\partial(\frac{1}{r_1})}{\partial r_1} = \frac{1}{r_1^2}, \quad r_1 = \varepsilon,$$

and consequently,

$$\int_{S^0_\varepsilon(y)} \frac{\partial_\xi(\frac{1}{r_1})}{\partial n_\xi}\, ds = \frac{1}{\varepsilon^2}\int_{S^0_\varepsilon(y)} ds = \omega_y(\Omega)$$

is the solid angle under which one can see Ω from the point y. Thus,

$$\int_\Omega \left|\frac{\partial_\xi(\frac{1}{r_1})}{\partial n_\xi}\right| ds = |\omega_y(\Omega)|.$$

Let now $\Sigma_\delta(x) = \Sigma^+_\delta(x) \cup \Sigma^-_\delta(x)$, and

$$\pm\frac{\partial_\xi(\frac{1}{r_1})}{\partial n_\xi} \ge 0$$

on $\Sigma_\delta^\pm(x)$. Then it can be shown that

$$\int_{\Sigma_\delta(x)} \left| \frac{\partial_\xi(\frac{1}{r_1})}{\partial n_\xi} \right| ds \le \int_{\Sigma_\delta^+(x)} \left| \frac{\partial_\xi(\frac{1}{r_1})}{\partial n_\xi} \right| ds + \int_{\Sigma_\delta^-(x)} \left| \frac{\partial_\xi(\frac{1}{r_1})}{\partial n_\xi} \right| ds \le C_* .$$

Lemma 4.6.6 is proved. □

Lemma 4.6.7. *The following relations are valid*

$$\int_\Sigma \frac{\partial_\xi(\frac{1}{r})}{\partial n_\xi} ds = \begin{cases} -4\pi, & x \in D, \\ -2\pi, & x \in \Sigma, \\ 0, & x \in D_1, \end{cases} \qquad r := \|x-\xi\|,\ D_1 := \mathbf{R}^3 \setminus \overline{D}. \tag{4.6.11}$$

Proof. 1) Let $x \in D_1$. Then the function $1/r$ is harmonic in $\overline{D}$ (with respect to ξ), and consequently, by Theorem 4.1.2,

$$\int_\Sigma \frac{\partial_\xi(\frac{1}{r})}{\partial n_\xi} ds = 0.$$

2) Let $x \in D$. In the domain $D \setminus K_\delta(x)$ the function $1/r$ is harmonic (with respect to ξ), and consequently, by Theorem 4.1.2,

$$\int_\Sigma \frac{\partial_\xi(\frac{1}{r})}{\partial n_\xi} ds + \int_{S_\delta(x)} \frac{\partial_\xi(\frac{1}{r})}{\partial n_\xi} ds = 0.$$

On $S_\delta(x)$:

$$\frac{\partial_\xi(\frac{1}{r})}{\partial n_\xi} = -\frac{\partial(\frac{1}{r})}{\partial r} = \frac{1}{r^2}, \quad r = \delta,$$

hence

$$\int_{S_\delta(x)} \frac{\partial_\xi(\frac{1}{r})}{\partial n_\xi} ds = \int_{S_\delta(x)} \frac{ds}{r^2} ds = \frac{1}{\delta^2} \int_{S_\delta(x)} ds = 4\pi,$$

and (4.6.11) is proved.

3) Let $x \in \Sigma$. In the domain $D \setminus K_\delta(x)$ the function $1/r$ is harmonic with respect to ξ, and consequently, by Theorem 4.1.2,

$$\int_{\Sigma\setminus\Sigma_\delta(x)} \frac{\partial_\xi(\frac{1}{r})}{\partial n_\xi} ds + \int_{S_\delta^0(x)} \frac{\partial_\xi(\frac{1}{r})}{\partial n_\xi} ds = 0,$$

where $S_\delta^0(x) = S_\delta(x) \cap \overline{D}$. Since

$$\int_{S_\delta^0(x)} \frac{\partial_\xi(\frac{1}{r})}{\partial n_\xi} ds = \int_{S_\delta^0(x)} \frac{ds}{r^2} = \frac{1}{\delta^2} \int_{S_\delta^0(x)} ds \to 2\pi \quad \text{for} \quad \delta \to 0,$$

we have, in view of (4.6.7),

$$\int_{\Sigma_\delta(x)} \frac{\partial_\xi(\frac{1}{r})}{\partial n_\xi} ds \to 0 \quad \text{for} \quad \delta \to 0,$$

and therefore we arrive at (4.6.11). Lemma 4.6.7 is proved. □

3. Properties of single- and double-layer potentials

We consider the functions $Q(x)$ and $F(x)$ of the form (4.6.1).

Theorem 4.6.1. *The functions $Q(x)$ and $F(x)$ are harmonic in D and D_1, and*

$$Q(x)=O\left(\frac{1}{\|x\|}\right),\quad F(x)=O\left(\frac{1}{\|x\|^2}\right)$$

as $x\to\infty$.

Proof. 1) Let $\overline{G}\subset D$ (or $\overline{G}\subset D_1$) be a bounded closed domain. Then for all $x\in\overline{G}$, $\xi\in\Sigma$ we have $r=\|x-\xi\|\ge d>0$, where d is the distance of $\overline{G}$ and Σ. Therefore, $Q(x)\in C^\infty(\overline{G})$, $F(x)\in C^\infty(\overline{G})$, and all their derivatives can be obtained by differentiation under the sign of integration. In particular,

$$\Delta Q(x)=\int_\Sigma q(\xi)\Delta\left(\frac{1}{r}\right)ds=0,$$

$$\Delta F(x)=\int_\Sigma f(\xi)\frac{\partial_\xi}{\partial n_\xi}\left(\Delta\left(\frac{1}{r}\right)\right)ds=0,$$

i.e. $Q(x)$ and $F(x)$ are harmonic in $\overline{G}$. By virtue of the arbitrariness of $\overline{G}$ we obtain that $Q(x)$ and $F(x)$ are harmonic in D and D_1.

2) By Lemma 4.3.1,

$$\frac{\partial_\xi r}{\partial n_\xi}=\cos(n_\xi,\bar r),$$

and consequently, the function $F(x)$ takes the form

$$F(x)=-\int_\Sigma f(\xi)\,\frac{\cos(n_\xi,\bar r)}{r^2}\,ds. \tag{4.6.12}$$

Let $D\subset K_R(0)$, $\|x\|\ge 2R$, $\xi\in\Sigma$. Then

$$r\ge\|x\|-\|\xi\|\ge\|x\|-R\ge\|x\|/2,$$

i.e. $1/r\le 2/\|x\|$. From (4.6.1) and (4.6.12) we obtain the estimates

$$|Q(x)|\le\frac{C}{\|x\|},\ |F(x)|\le\frac{C}{\|x\|^2},\ \|x\|\ge 2R.$$

Theorem 4.6.1 is proved. □

Theorem 4.6.2. *The function $F(x)$ exists and is continuous on Σ.*

Proof. Let $x\in\Sigma$, $\delta\le\delta_0$. It follows from (4.6.1) and Lemma 4.6.4 that

$$|F(x)|\le C_f\left(\int_{\Sigma_\delta(x)}\left|\frac{\partial_\xi(\frac{1}{r})}{\partial n_\xi}\right|ds+\int_{\Sigma\setminus\Sigma_\delta(x)}\left|\frac{\partial_\xi(\frac{1}{r})}{\partial n_\xi}\right|ds\right)$$

$$\le C_f\left(4\pi\Phi_0\delta+\int_{\Sigma\setminus\Sigma_\delta(x)}\frac{|\cos(\bar n_\xi,\bar r)|}{r^2}ds\right)\le C_f\left(4\pi\Phi_0\delta+\frac{S_\Sigma}{\delta^2}\right),$$

where $C_f = \sup_{\xi\in\Sigma} |f(\xi)|$ and S_Σ is the measure of Σ. Therefore, the integral in (4.6.1) converges absolutely and uniformly on Σ, and consequently, the function $F(x)$ exists and is bounded on Σ.

Let $y \in \Sigma_{\delta/2}(x) := K_{\delta/2}(x) \cap \Sigma$. We have

$$F(x) - F(y) = \int_\Sigma f(\xi) \left(\frac{\partial_\xi(\frac{1}{r})}{\partial n_\xi} - \frac{\partial_\xi(\frac{1}{r_1})}{\partial n_\xi} \right) ds,\ r := \|x-\xi\|,\ r_1 := \|y-\xi\|.$$

Using Lemma 4.6.4, we infer

$$\left| \int_{\Sigma_{\delta/2}(x)} f(\xi) \left(\frac{\partial_\xi(\frac{1}{r})}{\partial n_\xi} - \frac{\partial_\xi(\frac{1}{r_1})}{\partial n_\xi} \right) ds \right|$$
$$\le C_f \left(\int_{\Sigma_{\delta/2}(x)} \left| \frac{\partial_\xi(\frac{1}{r})}{\partial n_\xi} \right| ds + \int_{\Sigma_{\delta/2}(x)} \left| \frac{\partial_\xi(\frac{1}{r_1})}{\partial n_\xi} \right| ds \right)$$
$$\le C_f \left(\int_{\Sigma_{\delta/2}(x)} \left| \frac{\partial_\xi(\frac{1}{r})}{\partial n_\xi} \right| ds + \int_{\Sigma_{\delta}(y)} \left| \frac{\partial_\xi(\frac{1}{r_1})}{\partial n_\xi} \right| ds \right) \le C_0\delta,\ C_0 := 6\pi\Phi_0 C_f.$$

Take $\varepsilon > 0$. Let $\delta = \varepsilon/(2C_0)$. Then

$$\left| \int_{\Sigma_{\delta/2}(x)} f(\xi) \left(\frac{\partial_\xi(\frac{1}{r})}{\partial n_\xi} - \frac{\partial_\xi(\frac{1}{r_1})}{\partial n_\xi} \right) ds \right| \le \frac{\varepsilon}{2} \quad \text{for all} \quad y \in \Sigma_{\delta/2}(x).$$

On $\Sigma \setminus \Sigma_{\delta/2}(x)$ the integrand has no singularities. By virtue of the continuity, one has that $\exists\, \delta_1\ (\delta_1 < \delta/2) \quad \forall\, y \in \Sigma_{\delta_1}(x)$

$$\left| \int_{\Sigma\setminus\Sigma_{\delta/2}(x)} f(\xi) \left(\frac{\partial_\xi(\frac{1}{r})}{\partial n_\xi} - \frac{\partial_\xi(\frac{1}{r_1})}{\partial n_\xi} \right) ds \right| \le \frac{\varepsilon}{2}.$$

Thus, we have proved that

$$\forall\, \varepsilon > 0 \quad \exists\, \delta_1 \quad \forall\, y \in \Sigma_{\delta_1}(x) \quad |F(x) - F(y)| \le \varepsilon,$$

and consequently, $F(x) \in C(\Sigma)$. Theorem 4.6.2 is proved. □

Theorem 4.6.3. *The function $Q(x)$ exists and is continuous for all $x \in \mathbf{R}^3$.*

Proof. The case $x \notin \Sigma$ was considered in Theorem 4.6.1. Let $x \in \Sigma$, $\delta \le \delta_0$. It follows from (4.6.1) and Lemma 4.6.3 that

$$|Q(x)| \le C_q \left(\int_{\Sigma_\delta(x)} \frac{ds}{r} + \int_{\Sigma\setminus\Sigma_\delta(x)} \frac{ds}{r} \right) \le C_q \left(4\pi\delta + \frac{C_\Sigma}{\delta} \right).$$

Therefore, the integral in (4.6.1) converges absolutely and uniformly, and consequently, the function $Q(x)$ exists and is bounded on Σ. Let $y \in K_{\delta/2}(x)$. We have

$$|Q(x) - Q(y)| \le$$

$$C_q\left(\int_{\Sigma_{\delta/2}(x)}\left|\frac{1}{r}-\frac{1}{r_1}\right|ds+\int_{\Sigma\setminus\Sigma_{\delta/2}(x)}\left|\frac{1}{r}-\frac{1}{r_1}\right|ds\right),$$

where $r_1=\|y-\xi\|$. By virtue of Lemma 4.6.3,

$$\int_{\Sigma_{\delta/2}(x)}\frac{ds}{r}\le 2\pi\delta,\quad \int_{\Sigma_{\delta/2}(x)}\frac{ds}{r_1}\le 4\pi\delta.$$

Fix $\varepsilon>0$. Take $\delta=\varepsilon/(12\pi C_q)$. Then

$$C_q\int_{\Sigma_{\delta/2}(x)}\left|\frac{1}{r}-\frac{1}{r_1}\right|ds\le\frac{\varepsilon}{2}\quad\text{for all}\quad y\in K_{\delta/2}(x).$$

On $\Sigma\setminus\Sigma_{\delta/2}(x)$ the integrand has no singularities; hence by virtue of the continuity,

$$\exists\,\delta_1\ (\delta_1<\delta/2)\quad\forall\,y\in K_{\delta_1}(x)\quad C_q\int_{\Sigma\setminus\Sigma_{\delta/2}(x)}\left|\frac{1}{r}-\frac{1}{r_1}\right|ds\le\frac{\varepsilon}{2}.$$

Thus,

$$\forall\,\varepsilon>0\quad\exists\,\delta_1\quad\forall\,y\in K_{\delta_1}(x)\quad|Q(x)-Q(y)|\le\varepsilon.$$

Theorem 4.6.3 is proved. □

Denote

$$\Phi(x)=\int_\Sigma q(\xi)\frac{\partial_x(\frac{1}{r})}{\partial n_x}\,ds,\quad x\in\Sigma,\quad r=\|x-\xi\|. \tag{4.6.13}$$

Theorem 4.6.4. *The function* $\Phi(x)$ *exists and is continuous on* Σ.

We omit the proof since it is similar to the one of Theorem 4.6.2.

Theorem 4.6.5. *Let* $x\in\Sigma$. *Then there exist the finite limits*

$$F_+(x):=\lim_{y\to x,y\in D_1}F(y),\quad F_-(x):=\lim_{y\to x,y\in D}F(y),$$

and

$$F_\pm(x)=F(x)\pm 2\pi f(x),\quad x\in\Sigma. \tag{4.6.14}$$

Proof. Fix $x\in\Sigma$ and denote

$$J(y):=\int_\Sigma(f(\xi)-f(x))\,\frac{\partial_\xi(\frac{1}{r_1})}{\partial n_\xi}\,ds,\quad r_1:=\|y-\xi\|.$$

Then

$$F(y)=\int_\Sigma f(\xi)\frac{\partial_\xi(\frac{1}{r_1})}{\partial n_\xi}\,ds=J(y)+f(x)\int_\Sigma\frac{\partial_\xi(\frac{1}{r})}{\partial n_\xi}\,ds,$$

and consequently, by virtue of Lemma 4.6.7,

$$F(y)=\begin{cases} J(y)-4\pi f(x), & y\in D,\\ J(y), & y\in D_1,\end{cases} \tag{4.6.15}$$

Moreover, by Lemma 4.6.7,

$$J(x) = F(x) + 2\pi f(x), \quad x \in \Sigma. \tag{4.6.16}$$

Let us show that

$$\lim_{y \to x} J(y) = J(x). \tag{4.6.17}$$

We have

$$J(y) - J(x) = \int_\Sigma (f(\xi) - f(x)) \left(\frac{\partial_\xi(\frac{1}{r_1})}{\partial n_\xi} - \frac{\partial_\xi(\frac{1}{r})}{\partial n_\xi} \right) ds. \tag{4.6.18}$$

Let $\delta \le \delta_0, y \in K_{\delta/2}(x)$. Using Lemmas 4.6.4 and 4.6.6 we calculate

$$\int_{\Sigma_\delta(x)} \left| \frac{\partial_\xi(\frac{1}{r_1})}{\partial n_\xi} \right| ds + \int_{\Sigma_\delta(x)} \left| \frac{\partial_\xi(\frac{1}{r})}{\partial n_\xi} \right| ds \le C_* + 4\pi\Phi_0\delta \le C_* + \frac{\pi}{2} := C_1.$$

Fix $\varepsilon > 0$. Since $f \in C(\Sigma)$, there exists δ $(\delta \le \delta_0)$ such that $|f(\xi) - f(x)| \le \varepsilon/(2C_1)$ for all $\xi \in \Sigma_\delta(x)$. Then

$$\left| \int_{\Sigma_\delta(x)} (f(\xi) - f(x)) \left(\frac{\partial_\xi(\frac{1}{r_1})}{\partial n_\xi} - \frac{\partial_\xi(\frac{1}{r})}{\partial n_\xi} \right) ds \right| \le \frac{\varepsilon}{2} \quad \text{for all} \quad y \in K_{\delta/2}(x).$$

On $\Sigma \setminus \Sigma_\delta(x)$ the integrand in (4.6.18) has no singularities. Hence, by virtue of the continuity, $\exists\, \delta_1$ $(\delta_1 < \delta/2)$ $\quad \forall\, y \in K_{\delta_1}(x)$:

$$\left| \int_{\Sigma \setminus \Sigma_\delta(x)} (f(\xi) - f(x)) \left(\frac{\partial_\xi(\frac{1}{r_1})}{\partial n_\xi} - \frac{\partial_\xi(\frac{1}{r})}{\partial n_\xi} \right) ds \right|$$

$$\le 2C_f \int_{\Sigma \setminus \Sigma_\delta(x)} \left| \frac{\partial_\xi(\frac{1}{r_1})}{\partial n_\xi} - \frac{\partial_\xi(\frac{1}{r})}{\partial n_\xi} \right| ds \le \frac{\varepsilon}{2}.$$

Thus,

$$\forall\, \varepsilon > 0 \quad \exists\, \delta_1 \quad \forall\, y \in K_{\delta_1}(x) \quad |J(y) - J(x)| \le \varepsilon,$$

i.e. (4.6.17) is proved. All assertions of the theorem follow from (4.6.15)-(4.6.17). □

Theorem 4.6.6. *Let $x \in \Sigma$. Then there exist the finite limits*

$$\Phi_\pm(x) = \lim \frac{\partial Q(y)}{\partial n_x}, \quad y \to x,\ y = x \pm \alpha n_x,\ \alpha > 0,$$

(i.e. $Q(x)$ has on Σ normal derivatives from outside, denoted by $\Phi_+(x)$, and from inside, denoted by $\Phi_-(x)$). Moreover,

$$\Phi_\pm(x) = \Phi(x) \mp 2\pi q(x). \tag{4.6.19}$$

Proof. Fix $x \in \Sigma$ and denote

$$J_1(y) := \int_\Sigma q(\xi) \left(\frac{\partial_y(\frac{1}{r_1})}{\partial n_x} + \frac{\partial_\xi(\frac{1}{r_1})}{\partial n_\xi} \right) ds,\ r_1 := \|y - \xi\|. \tag{4.6.20}$$

Let us show that

$$\lim J_1(y) = J_1(x), \quad y \to x,\ y = x \pm \alpha n_x,\ \alpha > 0. \tag{4.6.21}$$

Let $\delta \le \delta_0, y \in K_{\delta/2}(x)$. We infer from (4.6.20) that

$$J_1(y) - J_1(x)$$

$$= \int_\Sigma q(\xi) \left(\left(\frac{\partial_y(\frac{1}{r_1})}{\partial n_x} + \frac{\partial_\xi(\frac{1}{r_1})}{\partial n_\xi} \right) - \left(\frac{\partial_x(\frac{1}{r})}{\partial n_x} + \frac{\partial_\xi(\frac{1}{r})}{\partial n_\xi} \right) \right) ds$$

$$= \int_{\Sigma_\delta(x)} + \int_{\Sigma \setminus \Sigma_\delta(x)}.$$

It follows from Lemmata 4.6.4-4.6.5 that

$$\left| \int_{\Sigma_\delta(x)} \right| \le C_0 \delta.$$

Fix $\varepsilon > 0$. Take $\delta = \varepsilon/(2C_0)$. Then

$$\left| \int_{\Sigma_\delta(x)} \right| \le \frac{\varepsilon}{2}.$$

On $\Sigma \setminus \Sigma_\delta(x)$ the integrand has no singularities. Hence, by virtue of the continuity,

$$\exists\, \delta_1\ (\delta_1 < \delta/2) \quad \forall\, y \in K_{\delta_1}(x) \quad \left| \int_{\Sigma \setminus \Sigma_\delta(x)} \right| \le \frac{\varepsilon}{2}.$$

Thus,

$$\forall\, \varepsilon > 0 \quad \exists\, \delta_1 \quad \forall\, y \in K_{\delta_1}(x) \quad |J_1(y) - J_1(x)| \le \varepsilon,$$

i.e. (4.6.21) is proved. Denote

$$Q_1(x) := \int_\Sigma q(\xi) \frac{\partial_\xi(\frac{1}{r})}{\partial n_\xi} ds, \quad r := \|x - \xi\|.$$

This is the double-layer potential with the density $q(\xi)$. By Theorem 4.6.5, for $x \in \Sigma$ there exist the finite limits

$$Q_1^\pm(x) := \lim Q_1(y), \quad y \to x, y = x \pm \alpha n_x, \alpha > 0,$$

and

$$Q_1^\pm(x) = Q_1(x) \pm 2\pi q(x), \quad x \in \Sigma. \tag{4.6.22}$$

We rewrite (4.6.20) in the form

$$\frac{\partial Q(y)}{\partial n_x} = J_1(y) - Q_1(y).$$

For $y \to x, y = x \pm \alpha n_x, \alpha > 0,$ we get, in view of (4.6.21)-(4.6.22),

$$\Phi_\pm(x) = J_1(x) - Q_1^\pm(x) = J_1(x) - Q_1(x) \mp 2\pi q(x).$$

Moreover, it follows from (4.6.20) that $J_1(x) = \Phi(x) + Q_1(x)$, $x \in \Sigma$, and we arrive at (4.6.21). Theorem 4.6.6 is proved. □

4. Basic results from the theory of Fredholm integral equations

Let $G \subset \mathbf{R}^n$ be a bounded closed set. We consider the following integral equation:

$$y(x) = \varphi(x) + \int_G K(x,\xi)y(\xi)\,d\xi, \quad x \in G, \tag{4.6.23}$$

where $\varphi(x) \in C(G)$, and the real kernel $K(x,\xi)$ satisfies the conditions

$$\int_G |K(x,\xi)|\,d\xi \le C, \quad \int_G K(x,\xi)y(\xi)\,d\xi \in C(G) \quad \forall\, y \in C(G).$$

A solution of (4.6.23) is a function $y(x) \in C(G)$, satisfying (4.6.23). Together with (4.6.23) we consider the following equations

$$y(x) = \int_G K(x,\xi)y(\xi)\,d\xi, \tag{$4.6.23_0$}$$

$$z(x) = \psi(x) + \int_G K(\xi,x)z(\xi)\,d\xi, \tag{$4.6.23^*$}$$

$$z(x) = \int_G K(\xi,x)z(\xi)\,d\xi. \tag{$4.6.23_0^*$}$$

Equation ($4.6.23^*$) is called conjugate to (4.6.23), and equations ($4.6.23_0$) and ($4.6.23_0^*$) are the corresponding homogeneous equations. The following Fredholm theorem is valid (see, for example, [3, Chapter 18] or [Freiling: Lectures on functional analysis, University of Duisburg, 2003]).

Fredholm's alternative theorem. 1) *If the homogeneous equation* $(4.6.23_0)$ *has only the trivial solution* $(y \equiv 0)$, *then equation* $(4.6.23)$ *has a unique solution for each* $\varphi(x) \in C(G)$.
2) *The homogeneous equations* $(4.6.23_0)$ *and* $(4.6.23_0^*)$ *have the same number of linear independent solutions.*
3) *Equation* $(4.6.23)$ *has a solution if and only if* $\int_G \varphi(\xi)z(\xi)\,d\xi = 0$ *for all solutions of equation* $(4.6.23_0^*)$.

5. Solution of the Dirichlet and Neumann problems

We consider the following problems

$$\Delta u = 0 \quad (x \in D), \quad u_{|\Sigma} = \varphi_-(x), \tag{A_-}$$

$$\Delta u = 0 \quad (x \in D_1), \quad u_{|\Sigma} = \varphi_+(x),\ u(\infty) = 0, \tag{A_+}$$

$$\Delta u = 0 \quad (x \in D), \quad \frac{\partial u}{\partial n}\Big|_\Sigma = \psi_-(x), \tag{B_-}$$

$$\Delta u = 0 \quad (x \in D_1), \quad \frac{\partial u}{\partial n}\Big|_\Sigma = \psi_+(x),\ u(\infty) = 0. \tag{B_+}$$

The problems (A_-) and (A_+) are the Dirichlet problems (interior and exterior, respectively), and the problems (B_-) and (B_+) are the Neumann problems (interior and exterior, respectively).

We will seek solutions of the Dirichlet problems (A_-) and (A_+) in the form of a double-layer potential:

$$F(x) = \int_\Sigma f(\xi) \frac{\partial_\xi(\frac{1}{r})}{\partial n_\xi} ds, \quad r := \|x - \xi\|, \quad f \in C(\Sigma). \tag{4.6.24}$$

By Theorem 4.6.1, the function $F(x)$ is harmonic in D and D_1, $F(\infty) = 0$, and by Theorem 4.6.5, it has finite limits $F_+(x)$ and $F_-(x)$ on Σ from outside and inside, respectively. Therefore, for the function $F(x)$ to be a solution of the problem $(A_\pm)$ it is necessary that $F_\pm(x) = \varphi_\pm(x)$. Then (4.6.14) takes the form:

$$\text{For } (A_-): \ \varphi_-(x) = F(x) - 2\pi f(x) \quad \text{or}$$

$$f(x) = -\frac{\varphi_-(x)}{2\pi} + \frac{1}{2\pi} \int_\Sigma f(\xi) \frac{\partial_\xi(\frac{1}{r})}{\partial n_\xi} ds, \ x \in \Sigma.$$

$$\text{For } (A_+): \ \varphi_+(x) = F(x) + 2\pi f(x) \quad \text{or}$$

$$f(x) = \frac{\varphi_+(x)}{2\pi} - \frac{1}{2\pi} \int_\Sigma f(\xi) \frac{\partial_\xi(\frac{1}{r})}{\partial n_\xi} ds, \ x \in \Sigma.$$

Denote

$$K(x,\xi) := \frac{1}{2\pi} \frac{\partial_\xi(\frac{1}{r})}{\partial n_\xi} = -\frac{1}{2\pi} \frac{\cos(\bar{n}_\xi, \bar{r})}{r^2},$$

$$\varphi_1(x) := -\frac{\varphi_-(x)}{2\pi}, \quad \varphi_2(x) := \frac{\varphi_+(x)}{2\pi}.$$

By virtue of Lemma 4.6.4 and Theorem 4.6.2,

$$\int_\Sigma |K(x,\xi)| \, ds \le C, \ \int_\Sigma K(x,\xi) f(\xi) \, ds \in C(\Sigma)$$

for any $f \in C(\Sigma)$. As a result we arrive at the following integral equations:

$$f(x) = \varphi_1(x) + \int_\Sigma K(x,\xi) f(\xi) \, ds, \ x \in \Sigma \quad (\text{for } (A_-)), \tag{4.6.25}$$

$$f(x) = \varphi_2(x) - \int_\Sigma K(x,\xi) f(\xi) \, ds, \ x \in \Sigma \quad (\text{for } (A_+)). \tag{4.6.26}$$

Thus, we have proved the following theorem.

Theorem 4.6.7. 1) *If $f(x)$ is a solution of equation* (4.6.25)*, then $F(x)$, defined by* (4.6.24)*, is a solution of problem (A_-).*
2) *If $f(x)$ is a solution of equation (4.6.26), then $F(x)$, defined by,* (4.6.24) *is a solution of problem (A_+).*

We will seek solutions of the Neumann problem (B_-) and (B_+) in the form of a single-layer potential:

$$Q(x) = \int_\Sigma q(\xi)\frac{1}{r}\,ds, \quad r := \|x-\xi\|, \quad q \in C(\Sigma). \tag{4.6.27}$$

By Theorem 4.6.1, the function $Q(x)$ is harmonic in D and D_1, $Q(\infty)=0$, and by Theorem 4.6.6, it has on Σ normal derivatives $\Phi_+(x)$ (from outside) and $\Phi_-(x)$ (from inside). Therefore, for the function $Q(x)$ to be a solution of problem $(B_\pm)$ it is necessary that $\Phi_\pm(x) = \psi_\pm(x)$. Then relations (4.6.19) take the form:

$$\text{For } (B_-): \ \psi_-(x) = \Phi(x) + 2\pi q(x) \quad \text{or}$$

$$q(x) = \frac{\psi_-(x)}{2\pi} - \frac{1}{2\pi}\int_\Sigma q(\xi)\frac{\partial_x(\frac{1}{r})}{\partial n_x}\,ds,\ x\in\Sigma,$$

$$\text{For } (B_+): \ \psi_+(x) = \Phi(x) - 2\pi q(x) \quad \text{or}$$

$$q(x) = -\frac{\psi_+(x)}{2\pi} + \frac{1}{2\pi}\int_\Sigma q(\xi)\frac{\partial_x(\frac{1}{r})}{\partial n_x}\,ds,\ x\in\Sigma.$$

Denote

$$\psi_1(x) := \frac{\psi_-(x)}{2\pi}, \quad \psi_2(x) := -\frac{\psi_+(x)}{2\pi}.$$

Since

$$\frac{1}{2\pi}\frac{\partial_x(\frac{1}{r})}{\partial n_x} = K(\xi,x),$$

we arrive at the following integral equations:

$$q(x) = \psi_1(x) - \int_\Sigma K(\xi,x)q(\xi)\,ds, \quad x\in\Sigma \qquad (\text{for } (B_-)), \tag{4.6.26*}$$

$$q(x) = \psi_2(x) + \int_\Sigma K(\xi,x)q(\xi)\,ds, \quad x\in\Sigma \qquad (\text{for } (B_+)). \tag{4.6.25*}$$

By virtue of Lemma 4.6.4 and Theorem 4.6.4,

$$\int_\Sigma |K(\xi,x)|\,ds \le C, \quad \int_\Sigma K(\xi,x)q(\xi)\,ds \in C(\Sigma)$$

for any $q\in C(\Sigma)$. Thus, we have proved the following theorem.

Theorem 4.6.8. 1) *If $q(x)$ is a solution of equation* (4.6.26*)*, then $Q(x)$, defined by* (4.6.27)*, is a solution of problem (B_-).*
2) *If $q(x)$ is a solution of equation (* 4.6.25* *), then $Q(x)$, defined by* (4.6.27)*, is a solution of problem (B_+).*

Let us study the integral equations (4.6.25), (4.6.26), (4.6.25*) and (4.6.26*).

Theorem 4.6.9. *For any continuous functions $\varphi_1(x)$ and $\psi_2(x)$ the solutions of the integral equations* (4.6.25) *and* (4.6.25*) *exist and are unique.*

Proof. Let $\tilde{q}(x)$ be a solution of the homogeneous equation

$$\tilde{q}(x) = \int_\Sigma K(\xi,x)\tilde{q}(\xi)\,ds, \Leftrightarrow \tilde{q}(x) = \frac{1}{2\pi}\int_\Sigma \tilde{q}(\xi)\frac{\partial_x(\frac{1}{r})}{\partial n_x}\,ds,\ x\in\Sigma. \qquad (4.6.25_0^*)$$

We consider the single-layer potential

$$\tilde{Q}(x) := \int_\Sigma \tilde{q}(\xi)\,\frac{1}{r}\,ds$$

and the corresponding functions $\tilde{\Phi}(x)$ and $\tilde{\Phi}_\pm(x)$. Then ($4.6.25_0^*$) has the form $\tilde{\Phi}(x) - 2\pi\tilde{q}(x) = 0$. On the other hand, by Theorem 4.6.6, $\tilde{\Phi}(x) - 2\pi\tilde{q}(x) = \tilde{\Phi}_+(x)$, and consequently, $\tilde{\Phi}_+(x) = 0$, $x\in\Sigma$. By the uniqueness theorem for problem (B_+), we have $\tilde{Q}(x) = 0$, $x\in D_1$. According to Theorem 4.6.3, $\tilde{Q}(x)$ is continuous in $\mathbf{R}^3$, hence $\tilde{Q}(x) = 0$, $x\in\Sigma$. By the uniqueness theorem for problem (A_-), we have $\tilde{Q}(x) = 0$, $x\in D$. Therefore, $\tilde{Q}(x)\equiv 0$, $x\in\mathbf{R}^3$, and consequently, $\tilde{\Phi}_\pm(x) = 0$, $x\in\Sigma$. By Theorem 4.6.6, $\tilde{q}(x) = 0$, $x\in\Sigma$. Thus, the homogeneous equation ($4.6.25_0^*$) has only the trivial solution. Using the Fredholm theorem we obtain the assertions of Theorem 4.6.9. □

The following theorems are corollaries of Theorems 4.6.7-4.6.9 and the uniqueness theorem for the problems (A_-) and (B_+).

Theorem 4.6.10. *For any continuous function $\varphi_-(x)$ the solution of the interior Dirichlet problem (A_-) exists, is unique and has the form* (4.6.24)*, where the function $f(x)$ is the solution of the integral equation* (4.6.25).

Theorem 4.6.11. *For any continuous function $\psi_+(x)$ the solution of the exterior Neumann problem (B_+) exists, is unique and has the form* (4.6.27)*, where the function $q(x)$ is the solution of the integral equation* ($4.6.25^*$).

Studying the problems (A_+) and (B_-) is a more complicated task. We consider the homogeneous equations

$$\tilde{f}(x) = -\int_\Sigma K(x,\xi)\tilde{f}(\xi)\,ds, \Leftrightarrow \tilde{f}(x) = -\frac{1}{2\pi}\int_\Sigma \tilde{f}(\xi)\frac{\partial_\xi(\frac{1}{r})}{\partial n_\xi}\,ds,\ x\in\Sigma, \qquad (4.6.26_0)$$

$$\tilde{q}(x) = -\int_\Sigma K(\xi,x)\tilde{q}(\xi)\,ds, \Leftrightarrow \tilde{q}(x) = -\frac{1}{2\pi}\int_\Sigma \tilde{q}(\xi)\frac{\partial_x(\frac{1}{r})}{\partial n_x}\,ds,\ x\in\Sigma. \qquad (4.6.26_0^*)$$

By virtue of Lemma 4.6.7, the function $\tilde{f}(x)\equiv 1$ is a solution of equation ($4.6.26_0$). Then, by Fredholm's theorem, equation ($4.6.26_0^*$) also has a nontrivial solution $\tilde{q}(x)\not\equiv 0$. For this solution we consider the single-layer potential

$$\tilde{Q}(x) := \int_\Sigma \tilde{q}(\xi)\,\frac{1}{r}\,ds$$

and the corresponding functions $\tilde{\Phi}(x)$ and $\tilde{\Phi}_\pm(x)$. Equation ($4.6.26_0^*$) has the form $\tilde{\Phi}(x) + 2\pi\tilde{q}(x) = 0$. On the other hand, by Theorem 4.6.6, $\tilde{\Phi}(x) + 2\pi\tilde{q}(x) = \tilde{\Phi}_-(x)$, and consequently, $\tilde{\Phi}_-(x) = 0$, $x\in\Sigma$. By the uniqueness theorem for problem (B_-), we have

$\tilde{Q}(x) \equiv C$, $x \in \overline{D}$. Let us show that equation ($4.6.26_0^*$) has not two linearly independent solutions. Indeed, let $\tilde{q}_1(x)$ and $\tilde{q}_2(x)$ be solutions of ($4.6.26_0^*$). Then

$$\tilde{Q}_k(x) := \int_\Sigma \tilde{q}_k(\xi)\,\frac{1}{r}\,ds \equiv C_k,\; x \in \overline{D}.$$

Denote $\hat{q}(x) := C_2\tilde{q}_1(x) - C_1\tilde{q}_2(x)$. Then

$$\hat{Q}_k(x) := \int_\Sigma \hat{q}_k(\xi)\,\frac{1}{r}\,ds \equiv 0,\; x \in \overline{D},$$

and consequently, by virtue of the uniqueness theorem for the problem (A_+), $\hat{Q}(x) \equiv 0$ in $\mathbf{R}^3$. Then we have $\hat{\Phi}_\pm(x) = 0$ and $\hat{q}(x) = 0$, i.e. $C_2\tilde{q}_1(x) \equiv C_1\tilde{q}_2(x)$.

Thus, equation ($4.6.26_0^*$) has only one nontrivial solution (up to a multiplicative constant). Normalizing: Let $q_0(x) \not\equiv 0$ be a solution of equation ($4.6.26_0^*$) such that

$$Q_0(x) := \int_\Sigma q_0(\xi)\,\frac{1}{r}\,ds \equiv 1,\; x \in \overline{D}.$$

The potential $q_0(x)$ is called the Roben potential. By Fredholm's theorem, equation ($4.6.26_0$) also has only one nontrivial solution (up to a multiplicative constant) $\tilde{f}(x) \equiv 1$. Using Fredholm's theorem again we arrive at the following assertion.

Theorem 4.6.12. 1) *The integral equation* $(4.6.26^*)$ *has a solution if and only if*

$$\int_\Sigma \psi_1(\xi)\,ds = 0.$$

If $q_1(x)$ *and* $q_2(x)$ *are solutions of* $(4.6.26^*)$, *then* $q_1(x) - q_2(x) \equiv Cq_0(x)$, *where* $q_0(x)$ *is the Roben potential.*
2) *The integral equation* $(4.6.26)$ *has a solution if and only if*

$$\int_\Sigma \varphi_2(\xi)q_0(\xi)\,ds = 0,$$

where $q_0(\xi)$ *is the Roben potential. If* $f_1(x)$ *and* $f_2(x)$ *are solutions of* $(4.6.26)$, *then* $f_1(x) - f_2(x) \equiv C$.

The following theorem is a corollary of Theorem 4.6.8, 4.6.12 and 4.2.3.

Theorem 4.6.13. *The interior Neumann problem (* B_- *) has a solution if and only if*

$$\int_\Sigma \psi_-(\xi)\,ds = 0.$$

The solution is defined up to a constant summand. Any solution of (B_-) *has the form* $(4.6.27)$, *where the function* $q(x)$ *is a solution of the integral equation* $(4.6.26^*)$.

It remains to study the exterior Dirichlet problem (A_+). We reduced it to equation (4.6.26). It follows from Theorem 4.6.12 that equation (4.6.26) has either no solutions or an infinite number of solutions. But this contradicts to the uniqueness theorem for the problem (A_+). The reason is that we seek a solution of problem (A_+) in the form (4.6.24), but by Theorem 4.6.1,

$$F(x) = O\left(\frac{1}{\|x\|^2}\right), \quad x \to \infty.$$

Thus, we do not take into account solutions of the class $O\left(\frac{1}{\|x\|}\right)$. Therefore, we will act as follows.

Fix $x^0 \in D$. Let us seek a solution of the problem (A_+) in the form

$$u(x) = F(x) + \frac{\alpha}{\|x - x^0\|},$$

where $F(x)$ has the form (4.6.24), $\alpha = const$. Clearly, $u(x)$ is harmonic in D_1, $u(\infty) = 0$ and $u(x) \in C(\overline{D_1})$. Therefore, for the function $u(x)$ to be a solution of (A_+), it is necessary that $u_{|\Sigma} = \varphi_+$, i.e.

$$\varphi_+(x) = F_+(x) + \frac{\alpha}{\|x - x^0\|}, \quad x \in \Sigma.$$

Since $F_+(x) = F(x) + 2\pi f(x)$, we arrive at the equation:

$$f(x) = \left(\frac{\varphi_+(x)}{2\pi} - \frac{\alpha}{2\pi\|x - x^0\|}\right) - \int_\Sigma K(x,\xi) f(\xi)\, ds. \tag{4.6.28}$$

According to Theorem 4.6.12, equation (4.6.28) is solvable if and only if

$$\int_\Sigma \left(\frac{\varphi_+(\xi)}{2\pi} - \frac{\alpha}{2\pi\|\xi - x^0\|}\right) q_0(\xi)\, ds = 0.$$

Since $x^0 \in D$ and q_0 is the Roben potential, we have

$$\int_\Sigma q_0(\xi) \frac{1}{\|\xi - x^0\|}\, ds = 1,$$

and consequently,

$$\alpha = \int_\Sigma \varphi_+(\xi) q_0(\xi)\, ds.$$

Thus, we have proved the following assertion.

Theorem 4.6.14. *For any continuous function $\varphi_+(x)$ the solution of the exterior Dirichlet problem (A_+) exists, is unique and has the form*

$$u(x) = F(x) + \frac{1}{\|x - x^0\|} \int_\Sigma \varphi_+(\xi) q_0(\xi)\, ds, \quad x \in D_1,$$

where $q_0(x)$ is the Roben potential, $F(x)$ has the form (4.6.24), *and $f(x)$ is a solution of the integral equation* (4.6.28).

We note that equation (4.6.28) has an infinite number of solutions but $F(x)$ is unique. Indeed, if $\hat{f}(x)$ is another solution (different from $f(x)$) of equation (4.6.28), then $\hat{f}(x) - f(x) \equiv C$, and by virtue of Lemma 4.6.7,

$$\int_\Sigma \left(f(\xi) - \hat{f}(\xi)\right) \frac{\partial_\xi(\frac{1}{r})}{\partial n_\xi}\, ds = C \int_\Sigma \frac{\partial_\xi(\frac{1}{r})}{\partial n_\xi}\, ds = 0, \quad x \in D_1.$$

4.7. The Variational Method

1. The variational Dirichlet principle

Let $D \subset \mathbf{R}^n$ be a bounded domain with the boundary $\Sigma \in PC^1$. We consider the Dirichlet problem

$$\left.\begin{aligned} \Delta u = 0 \quad (x \in D), \\ u_{|\Sigma} = \varphi(x), \quad \varphi(x) \in C(\Sigma). \end{aligned}\right\} \tag{4.7.1}$$

Denote

$$R_\varphi := \{u : \; u \in C(\overline{D}), u \in C^2(D), u_{|\Sigma} = \varphi\}.$$

A function $u(x)$ is called a solution of problem (4.7.1) if $u \in R_\varphi$ and $\Delta u = 0$ in D. We consider the Dirichlet functional

$$\Phi(u) := \int_D \Big(\sum_{k=1}^n \Big(\frac{\partial u}{\partial x_k}\Big)^2 \Big)\, dx \tag{4.7.2}$$

and formulate the variational extremal problem

$$\Phi(u) \to \inf, \quad u \in R_\varphi. \tag{4.7.3}$$

Clearly, $\Phi(u) \geq 0$, and consequently, there exists $\inf\limits_{u \in R_\varphi} \Phi(u)$. Suppose that there exists a function $u^* \in R_\varphi$ such that

$$\Phi(u^*) = \inf_{u \in R_\varphi} \Phi(u).$$

Then u^* satisfies the Euler equation. For the functional (4.7.2) the Euler equation has the form $\Delta u = 0$. Therefore, if u^* is a solution of the variational problem (4.7.3), then u^* is a solution of problem (4.7.1). Thus, we replace (4.7.1) by the variational problem (4.7.3). This approach is fruitful since one can use methods of the variational calculus.

Criticism. 1) It can happen that $\inf \Phi(u) = \infty$, i.e. $\Phi(u) = \infty \;\; \forall\, u \in R_\varphi$. An example of such a function is due to Hadamard (see [8, Chapter 22]). In this case the variational method does not work (although a solution of the problem (4.7.1) can exist as in the Hadamard example). In order to apply the variational method we impose an additional condition. Denote $Q_\varphi := \{u \in R_\varphi : \; \Phi(u) < \infty\}$.

$$\text{Continuability (extendability) condition:} \qquad Q_\varphi \neq \emptyset. \tag{4.7.4}$$

For such φ we have $\inf \Phi(u) < \infty$. In particular, if Σ is smooth, and $\varphi \in C^1$, then condition (4.7.4) is fulfilled automatically.

2) We suppose that problem (4.7.3) has a solution. This is not always valid, i.e. the infimum in (4.7.3) is not always attained (see [8, Chapter 22]). However, if one takes simply connected domains and extends the notion of the solution, then the variational method gives us the existence and the uniqueness of the solution of the Dirichlet problem, and also a constructive procedure for its solution.

2. Operator equations in a Hilbert space

Let H be a real Hilbert space with the scalar product $(\cdot,\cdot)$ and with the norm $\|\cdot\|$. Let $A: D_A \to E_A$ be a symmetric operator in H with an everywhere dense domain D_A.

Definition 4.7.1. The operator A is called positive definite ($A > 0$), if $(Au,u) \ge \gamma \|u\|^2$ for some $\gamma > 0$ and for all $u \in D_A$.

Everywhere below we assume that $A > 0$. In particular, this yields that on E_A there exists the inverse operator A^{-1}. We consider in H the equation

$$Au = f, \quad f \in H. \tag{4.7.5}$$

An element u^* is called a solution of (4.7.5), if $u^* \in D_A$ and $Au^* = f$. The following theorem is obvious.

Theorem 4.7.1. 1) *If a solution of* (4.7.5) *exists, then it is unique.*
2) *A solution of* (4.7.5) *exists if and only if* $f \in E_A$. *In particular, if* $E_A = H$, *then a solution of* (4.7.5) *exists for all* $f \in H$.

Consider the functional

$$F(u) = (Au,u) - 2(u,f), \quad u \in D_A \tag{4.7.6}$$

on D_A and the extremal problem

$$F(u) \to \inf, \quad u \in D_A. \tag{4.7.7}$$

An element $u^* \in D_A$ is called a solution of the problem (4.7.7), if $F(u^*) = \inf\limits_{u \in D_A} F(u)$.

Theorem 4.7.2. *The problems (4.7.5) and (4.7.7) are equivalent, i.e. they are solvable or unsolvable simultaneously, and* u^* *is a solution of (4.7.5) if and only if* u^* *is a solution of (4.7.7).*

Proof. 1) Let $Au^* = f, u^* \in D_A$. Let $v \in D_A$, $\eta = v - u^*$, i.e. $v = u^* + \eta$. Then

$$F(v) = (A(u^* + \eta), u^* + \eta) - 2(u^* + \eta, f) = F(u^*) + (A\eta, \eta) > F(u^*),$$

and consequently, u^* is a solution of (4.7.7).

2) Let u^* be a solution of (4.7.7), and let $\eta \in D_A, \lambda = const.$. Then $u^* + \lambda\eta \in D_A$, and consequently, $F(u^* + \lambda\eta) \ge F(u^*)$. This yields $2\lambda(Au^* - f, \eta) + \lambda^2(A\eta, \eta) \ge 0$. This is possible only if $(Au^* - f, \eta) = 0$ for all $\eta \in D_A$. Since D_A is everywhere dense, we have $Au^* = f$, and Theorem 4.7.2 is proved. □

Remark 4.7.1. If $E_A \ne H, f \notin E_A$, then the problems (4.7.5) and (4.7.7) have no solutions. We extend the notion of the solution. For this purpose we introduce the so-called energy space.

We introduce a new scalar product and the corresponding norm: $[u,v] := (Au,v)$, $|u|^2 = [u,u]$, $u,v \in D_A$. Then we complete this space by taking its closure, i.e. by including the limit elements with respect to its norm. As a result we obtain a new Hilbert space which is denoted by H_A and is called the energy space. It can be shown [9] that $H_A \subset H$.

Example 4.7.1. Let $H = L_2(0,1)$,

$$Au = -u''(x),\ u(0) = u(1) = 0, \quad D_A = \{u \in C^2[0,1] :\ u(0) = u(1) = 0\}.$$

Then

$$(Au,u) = \int_0^1 \Big(u'(x)\Big)^2 dx.$$

Since

$$u(x) = \int_0^1 u'(x)\, dx,$$

we have $|u(x)| \le \|u'\|_{L_2}$, and consequently, $\|u\|_{L_2} \le \|u'\|_{L_2}$, i.e. $A > 0$. Moreover,

$$H_A = \{u(x) \in AC[0,1] :\ u'(x) \in L_2(0,1), u(0) = u(1) = 0\}.$$

We extend the domain of the functional $F(u)$ of the form (4.7.6) to H_A, namely, we consider the functional

$$F(u) = [u,u] - 2(u,f), \quad u \in H_A.$$

Since

$$|u|^2 = (Au,u) \ge \gamma \|u\|^2,$$

we have

$$|(u,f)| \le \|f\| \cdot \|u\| \le \gamma^{-1/2} \|f\| \cdot |u|,$$

and consequently, the functional (u,f) is bounded in H_A. By the Riesz representation theorem, there exists a unique element $f^* \in H_A$ such that $(u,f) = [u,f^*]$. Therefore the functional takes the form

$$F(u) = [u,u] - 2[u,f^*], \quad u \in H_A. \tag{4.7.8}$$

Consider the extremum problem

$$F(u) \to \inf, \quad u \in H_A. \tag{4.7.9}$$

An element $u^* \in H_A$ is called a solution of (4.7.9), if $F(u^*) = \inf\limits_{u \in H_A} F(u)$.

Theorem 4.7.3. *For each* $f^* \in H_A$ *problem* (4.7.9) *has a unique solution* u^*, *and* $u^* = f^*$.

Indeed, by virtue of (4.7.8),

$$F(u) = [u - f^*, u - f^*] - [f^*, f^*] = |u - f^*|^2 - |f^*|^2.$$

This yields the assertion of the theorem.

Definition 4.7.2. An element $u^* \in H_A$, which gives a solution of (4.7.9) (i.e. $u^* = f^*$), is called a generalized solution of problem (4.7.5).

One can give another (equivalent) definition of a generalized solution of problem (4.7.5). Indeed, from (4.7.5) we infer $(Au,v) = (f,v)$, $v \in H_A$ or $[u,v] = (f,v)$.

Definition 4.7.3. An element $u^* \in H_A$ is called a generalized solution of problem (4.7.5) if $[u^*, v] = (f, v)$ for all $v \in H_A$.

These definitions are equivalent since $(f, v) = [f^*, v]$.

Theorem 4.7.4. *For each* $f \in H$ *a generalized solution of problem* (4.7.5) *exists, is unique and has the form* $u^* = f^*$.

3. Solution of the Dirichlet problem

Next we apply the theory developed above for the solution of the Dirichlet problem. We confine ourselves to the cases $n = 2$ and $n = 3$. Let us consider the following Dirichlet problem:

$$\left.\begin{array}{c} -\Delta u = f \quad (x \in D), \quad f \in L_2(D), \\ u_{|\Sigma} = 0, \end{array}\right\} \tag{4.7.10}$$

where D is a bounded simply connected domain. Denote

$$R := \{u : \ u \in W_2^2(D), u_{|\Sigma} = 0\} =: W_2^{2,0}.$$

By the embedding theorem $u \in C(\overline{D})$.

Definition 4.7.4. A function u^* is called a(classical) solution of problem (4.7.10), if $u^* \in R$ and $-\Delta u^* = f$.

Problem (4.7.10) is a particular case of problem (4.7.5), where $H = L_2(D), A = -\Delta, D_A = R$. It is known [9] that $A > 0$. In this case

$$[u, v] = \int_D \left(\sum_{k=1}^{n} \frac{\partial u}{\partial x_k} \frac{\partial v}{\partial x_k} \right) dx.$$

One can show that $H_A = W_2^{1,0}$.

Definition 4.7.5. A function u^* is called a generalized solution of problem (4.7.10), if $u^* \in W_2^{1,0}$ and $[u^*, v] = (f, v)$ for all $v \in W_2^{1,0}$.

By virtue of Theorem 4.7.4, for any $f \in L_2(D)$ a generalized solution of problem (4.7.10) exists and is unique. We note that for (4.7.10) a generalized solution is also a classical one. In other words, the following assertion is valid [9]:

Theorem 4.7.5. *Let* $u^* \in W_2^{1,0}$ *be a generalized solution of* (4.7.10). *Then* $u^* \in W_2^{2,0}$ *and* $-\Delta u^* = f$, *i.e.* u^* *is a classical solution of* (4.7.10).

Corollary 4.7.1. *For any* $f \in L_2(D)$, *a classical solution of problem* (4.7.10) *exists and is unique.*

Now we consider the Dirichlet problem for the homogeneous equation:

$$\left.\begin{array}{c} \Delta u = 0 \quad (x \in D), \\ u_{|\Sigma} = \varphi(x), \quad \varphi(x) \in C(\Sigma). \end{array}\right\} \tag{4.7.11}$$

Denote $R_\varphi := \{u:\ u \in W_2^2(D), u_{|\Sigma} = \varphi\}$. If $u \in R_\varphi$, then by the embedding theorem $u \in C(\overline{D})$.

Definition 4.7.6. A function u^* is called a solution of problem (4.7.11) if $u^* \in R_\varphi$ and $\Delta u^* = 0$ in D.

Denote $Q_\varphi := \{u \in R_\varphi :\ \Phi(u) < \infty\}$, where $\Phi(u)$ is defined by (4.7.2). Let $Q_\varphi \neq \emptyset$, i.e. the continuability (extendability) condition (4.7.4) is fulfilled. Fix $w \in Q_\varphi$. Clearly, if $u \in R_\varphi$, then $v := u - w \in R$, and conversely, if $v \in R$, then $u := v + w \in R_\varphi$. This yields that the function u^* is a solution of problem (4.7.11) if and only if the function $v^* = u^* - w$ is a solution of the problem

$$\left.\begin{aligned} -\Delta v = f \quad (x \in D), \quad f := \Delta w \in L_2(D), \\ v_{|\Sigma} = 0. \end{aligned}\right\} \tag{4.7.12}$$

According to Corollary 4.7.1, the (classical) solution of problem (4.7.12) exists and is unique. Thus, we have proved the following assertion.

Theorem 4.7.6. *Let φ be such that $Q_\varphi \neq \emptyset$. Then a solution of the problem* (4.7.11) *exists and is unique.*

For more details on variational methods in connection with problems of mathematical physics we refer the reader to textbooks like [7], [8] and [9].

Chapter 5.

The Cauchy-Kowalevsky Theorem

Definition 5.1. A function $f(x_1,\dots,x_n)$ is called analytic at the point $(x_1^0,\dots,x_n^0)$, if in some neighbourhood of this point it can be represented in the form of an absolutely convergent power series

$$f(x_1,\dots,x_n)=\sum_{k_1,\dots,k_n=0}^{\infty} A_{k_1,\dots,k_n}(x_1-x_1^0)^{k_1}\dots(x_n-x_n^0)^{k_n}. \tag{5.1}$$

Consequently, the function f has partial derivatives of all orders and

$$A_{k_1,\dots,k_n}=\frac{1}{k_1!\dots k_n!}\frac{\partial^{k_1+\dots+k_n} f}{\partial x_1^{k_1}\dots\partial x_n^{k_n}}\Big|_{x_1=x_1^0,\dots,x_n=x_n^0}.$$

Moreover, the series (5.1) can be differentiated termwise in its domain of convergence an arbitrary number of times.

Without loss of generality we will consider the case when

$$(x_1^0,\dots,x_n^0)=(0,\dots,0).$$

Then (5.1) takes the form

$$f(x_1,\dots,x_n)=\sum_{k_1,\dots,k_n=0}^{\infty} A_{k_1,\dots,k_n}x_1^{k_1}\dots x_n^{k_n}. \tag{5.2}$$

Definition 5.2. A function $F(x_1,\dots,x_n)$ is called a *majorant* for a function $f(x_1,\dots,x_n)$ of the form (5.2), if $F(x_1,\dots,x_n)$ is analytic in a neighbourhood of the origin:

$$F(x_1,\dots,x_n)=\sum_{k_1,\dots,k_n=0}^{\infty} B_{k_1,\dots,k_n}x_1^{k_1}\dots x_n^{k_n} \tag{5.3}$$

in a neighbourhood of 0, and

$$|A_{k_1,\dots,k_n}|\le B_{k_1,\dots,k_n}$$

for all $k_1,\dots,k_n$. Clearly, the radius of convergence of (5.2) is not less than the radius of convergence of (5.3).

Lemma 5.1. *Let the function $f(t,x_1,\dots,x_n)$ be analytic at the origin:*

$$f(t,x_1,\dots,x_n)=\sum_{k_0,\dots,k_n=0}^{\infty}A_{k_0,\dots,k_n}t^{k_0}x_1^{k_1}\dots x_n^{k_n}. \tag{5.4}$$

Then there exist $M>0$ and $a>0$ such that for each fixed $\alpha\in(0,1]$ the function

$$F(t,x_1,\dots,x_n)=\frac{M}{1-\frac{t/\alpha+x_1+\dots+x_n}{a}} \tag{5.5}$$

is a majorant for f.

Proof. Since the series (5.4) converges absolutely in a neighbourhood of the origin, there exist positive numbers $a_0,a_1,\dots,a_n$ such that

$$\sum_{k_0,\dots,k_n=0}^{\infty}|A_{k_0,\dots,k_n}|a_0^{k_0}\dots a_n^{k_n}<\infty.$$

In particular, there exists $M>0$ such that

$$|A_{k_0,\dots,k_n}|a_0^{k_0}\dots a_n^{k_n}\le M$$

for all $k_1,\dots,k_n$. Therefore,

$$|A_{k_0,\dots,k_n}|\le\frac{M}{a_0^{k_0}\dots a_n^{k_n}}.$$

Let $a=\min\limits_{0\le m\le n}a_m$. Then

$$|A_{k_0,\dots,k_n}|\le\frac{M}{a^{k_0+\dots+k_n}}. \tag{5.6}$$

Consider a function $F(t,x_1,\dots,x_n)$ of the form (5.5) with these $M>0$ and $a>0$. Since (5.5) is the sum of terms of the geometric progression, we have for $|t|/\alpha+|x_1|+\dots+|x_n|<a$

$$F(t,x_1,\dots,x_n)=M\sum_{k=0}^{\infty}\frac{(t/\alpha+x_1+\dots+x_n)^k}{a^k}$$
$$=M\sum_{k=0}^{\infty}\frac{1}{a^k}\sum_{k_0+\dots+k_n=k}\frac{k!}{k_0!\dots k_n!}\left(\frac{t}{\alpha}\right)^{k_0}x_1^{k_1}\dots x_n^{k_n}.$$

Taking (5.6) into account we obtain

$$B_{k_0,\dots,k_n}=\frac{M}{a^{k_0+\dots+k_n}\alpha^{k_0}}\frac{(k_0+\dots+k_n)!}{k_0!\dots k_n!}\ge\frac{M}{a^{k_0+\dots+k_n}}\ge|A_{k_0,\dots,k_n}|,$$

and Lemma 5.1 is proved. □

The general Cauchy-Kowalevsky theorem is a fundamental theorem on the existence of the solution of the Cauchy problem for a wide class of systems of partial differential equations. For simplicity we confine ourselves here to the case of *linear* systems of the first order. We consider the following Cauchy problem:

$$\frac{\partial u_i}{\partial t}=\sum_{j=1}^{N}\sum_{k=1}^{n}a_{ijk}(t,x_1,\dots,x_n)\frac{\partial u_j}{\partial x_k}+\sum_{j=1}^{N}b_{ij}(t,x_1,\dots,x_n)u_j$$

$$+c_i(t,x_1,\ldots,x_n), \tag{5.7}$$

$$u_{i|t=t_0} = \varphi_i(x_1,\ldots,x_n), \quad i=\overline{1,N}, \tag{5.8}$$

Here $t,x_1,\ldots,x_n$ are independent variables and $u_1,\ldots,u_N$ are unknown functions.

Theorem 5.1. *Let the functions* a_{ijk}, b_{ij}, c_i *be analytic at the point* $(t_0,x_1^0,\ldots,x_n^0)$, *and let the functions* φ_i *be analytic at* $(x_1^0,\ldots,x_n^0)$. *Then in some neighbourhood of the point* $(t_0,x_1^0,\ldots,x_n^0)$ *the Cauchy problem* $(5.7)-(5.8)$ *has a unique analytic solution.*

Proof. Without loss of generality we assume that $t_0 = x_1^0 = \ldots = x_n^0 = 0$, i.e. we seek a solution in a neighbourhood of the origin. Without loss of generality we also assume that $\varphi_i(x_1,\ldots,x_n) \equiv 0$. Otherwise, for the functions $v_i := u_i - \varphi_i$ we obtain a system of the form (5.7) with different c_i, but with zero initial conditions. Thus, we will solve the Cauchy problem for system (5.7) in a neighbourhood of the origin with the initial conditions

$$u_i(0,x_1,\ldots,x_n) = 0. \tag{5.9}$$

Step 1: Constructing of the solution. Suppose that there exists an analytic (at the origin) solution of the Cauchy problem (5.7), (5.9):

$$u_i(t,x_1,\ldots,x_n) = \sum_{k_0,\ldots,k_n=0}^{\infty} \alpha^i_{k_0,\ldots,k_n} t^{k_0} x_1^{k_1} \ldots x_n^{k_n}, \tag{5.10}$$

$$\alpha^i_{k_0,\ldots,k_n} = \frac{1}{k_0!\ldots k_n!} \frac{\partial^{k_0+\ldots+k_n} u_i}{\partial t^{k_0} \partial x_1^{k_1} \ldots \partial x_n^{k_n}}_{|t=0,x_1=\ldots=x_n=0}.$$

Let us give a constructive procedure for finding the coefficients $\alpha^i_{k_0,\ldots,k_n}$.

1) Differentiating (5.9) k_1 times with respect to x_1, $\ldots$, k_n times with respect to x_n and taking $t = x_1 = \ldots = x_n = 0$, we obtain $\alpha^i_{0,k_1\ldots,k_n} = 0$.

2) We differentiate (5.7) k_1 times with respect to x_1, $\ldots$, k_n times with respect to x_n and take $t = x_1 = \ldots = x_n = 0$. Then from the left we obtain $\alpha^i_{1,k_1\ldots,k_n}$, and from the right we obtain known quantities. Thus, we have constructed the coefficients $\alpha^i_{1,k_1,\ldots,k_n}$.

3) We differentiate (5.7) once with respect to t, k_1 times with respect to $x_1,\ldots$, k_n times with respect to x_n and take $t = x_1 = \ldots = x_n = 0$. Then from the left we obtain $\alpha^i_{2,k_1\ldots,k_n}$, and from the right we obtain known quantities which depend on $\alpha^i_{1,k_1,\ldots,k_n}$. Thus, we have constructed the coefficients $\alpha^i_{2,k_1,\ldots,k_n}$.

4) We continue this process by induction. Suppose that the coefficients $\alpha^i_{j,k_1,\ldots,k_n}$ have been already constructed for $j=\overline{0,k_0-1}$. We differentiate (5.7) k_0-1 times with respect to t, k_1 times with respect to $x_1,\ldots$, k_n times with respect to x_n and take $t = x_1 = \ldots = x_n = 0$. Then from the left we obtain $\alpha^i_{k_0,k_1\ldots,k_n}$, and from the right we obtain known quantities which depend on $\alpha^i_{j,k_1,\ldots,k_n}$, $j=\overline{0,k_0-1}$. Thus, we have constructed the coefficients $\alpha^i_{k_0,k_1,\ldots,k_n}$.

This procedure of constructing the analytic solution (5.10) of the Cauchy problem (5.7), (5.9) is called the A-procedure. In particular, this yields the uniqueness of the analytic solution. It remains to show that the series (5.10), constructed by the A-procedure, converges in some neighbourhood of the origin.

Step 2: Proof of the convergence of the series (5.10). We use Lemma 5.1. Choose $M>0$ and $a>0$ such that a function F of the form (5.5) is simultaneously a majorant for all coefficients a_{ijk}, b_{ij}, c_i of system (5.7). We consider the so-called majorizing system

$$\frac{\partial v_i}{\partial t} = F(t,x_1,\ldots,x_n)\left(\sum_{j=1}^{N}\sum_{k=1}^{n}\frac{\partial v_j}{\partial x_k} + \sum_{j=1}^{N} v_j + 1\right), \tag{5.11}$$

which is obtained from (5.7) by replacing all coefficients a_{ijk}, b_{ij}, c_i by their majorant F. We will seek a solution of system (5.11) in the form

$$v_1 = \ldots = v_N = v(z), \quad \text{where} \quad z = \frac{t}{\alpha} + x_1 + \ldots + x_n. \tag{5.12}$$

Substituting (5.12) into (5.11) we obtain for $v(z)$ the following ordinary differential equation

$$\frac{1}{\alpha}\frac{dv}{dz} = F(z)\left(Nn\frac{dv}{dz} + Nv + 1\right),$$

where

$$F(z) := \frac{M}{1 - z/a}.$$

Therefore,

$$\frac{dv}{dz} = B(z)(Nv+1), \tag{5.13}$$

where

$$B(z) := \frac{F(z)}{1/\alpha - NnF(z)}.$$

Choose $\alpha \in (0,1]$ such that $1/\alpha - NnF(z) > 0$ in a neighbourhood of the point $z=0$. Then in this neighbourhood $B(z)$ is analytic. A particular solution of equation (5.13) has the form

$$v(z) = \frac{1}{N}\left(e^{D(z)} - 1\right), \quad D(z) := N\int_0^z B(\xi)\,d\xi. \tag{5.14}$$

The function $v(z)$ of the form (5.14) is analytic at the point $z=0$, and all its Taylor coefficients are positive. Indeed, the Taylor coefficients of $F(z)$ are positive:

$$F(z) = M\sum_{k=0}^{\infty}\frac{z^k}{a^k}.$$

Furthermore,

$$B(z) = \alpha F(z)\sum_{s=0}^{\infty}(\alpha NnF(z))^s,$$

and consequently, the Taylor coefficients of the function $B(z)$ are positive as well. Obviously, the functions $D(z)$ and $e^{D(z)} - 1$ possess the same property. Therefore, all the Taylor coefficients of the function $v(z)$ are positive.

Thus, system (5.11) has in a neighbourhood of the origin an analytic solution of the form

$$v_i(t,x_1,\ldots,x_n) = \sum_{k_0,\ldots,k_n}^{\infty}\tilde{\alpha}^i_{k_0,\ldots,k_n} t^{k_0} x_1^{k_1}\ldots x_n^{k_n}, \tag{5.15}$$

where $\tilde{\alpha}^i_{k_0,\dots,k_n} > 0$, and the series (5.15) converges absolutely in a neighbourhood of the origin.

On the other hand, the coefficients $\tilde{\alpha}^i_{k_0,\dots,k_n}$ can be calculated by the A-procedure, using system (5.11) and the initial conditions $v_{i|t=0} = v(x_1+\dots+x_n)$, in the same way as the coefficients $\alpha^i_{k_0,\dots,k_n}$ were calculated by the A-procedure, using (5.7) and the initial conditions (5.9). We note that for calculating the coefficients $\alpha^i_{k_0,\dots,k_n}$ and $\tilde{\alpha}^i_{k_0,\dots,k_n}$ in the A-procedure we use only the operations of addition and multiplication. Since F is a majorant for the coefficients of system (5.7), and the initial data are majorants for the initial data (5.9), we conclude that

$$|\alpha^i_{k_0,\dots,k_n}| \le \tilde{\alpha}^i_{k_0,\dots,k_n}.$$

Therefore, the series (5.10) converges absolutely in a neighbourhood of the origin and gives us the solution of the Cauchy problem (5.7), (5.9). Theorem 5.1 is proved. □

Remark 5.1. The Cauchy-Kowalevsky theorem is also valid for a wide class of nonlinear systems of the form

$$\left.\begin{aligned} \frac{\partial^{n_1} u_1}{\partial t^{n_1}} &= F_1\Big(t, x_1, \dots, x_n, u_1, \dots, u_N, \dots, \frac{\partial^k u_1}{\partial t^{k_0}\partial x_1^{k_1}\dots\partial x_n^{k_n}}, \dots\Big), \\ &\dots\dots\dots \\ \frac{\partial^{n_N} u_N}{\partial t^{n_N}} &= F_N\Big(t, x_1, \dots, x_n, u_1, \dots, u_N, \dots, \frac{\partial^k u_1}{\partial t^{k_0}\partial x_1^{k_1}\dots\partial x_n^{k_n}}, \dots\Big), \end{aligned}\right\} \tag{5.16}$$

where $k_0+\dots+k_n \le n_j, k_0 < n_j$. Such systems are called normal systems or the Kowalevsky systems. For example, the equation of a vibrating string

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

has the form (5.16), but the heat conduction equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

has not the form (5.16).

Remark 5.2. For systems, which have not the form (5.16), the Cauchy-Kowalevsky theorem, generally speaking, is not valid. Example:

$$\left.\begin{aligned} u_t &= u_{xx}, \\ u_{|t=0} &= \frac{1}{1-x}, \quad |x| < 1. \end{aligned}\right\} \tag{5.17}$$

If an analytic solution of the Cauchy problem (5.17) exists, then it must have the form

$$\sum_{n=0}^{\infty} \frac{(2n)!t^n}{n!(1-x)^{2n+1}}.$$

However, for $t \ne 0$ this series is divergent.

Chapter 6.

Exercises

This chapter contains exercises for the course ”Differential Equations of Mathematical Physics”. The material here reflects all main types of equations of mathematical physics and represents the main methods for the solution of these equations.

6.1. Classification of Second-Order Partial Differential Equations

In order to solve the exercises in this section use the theory from Section 1.2.

6.1.1. Reduce the following equations to the canonical form in each domain where the equation preserves its type:

1. $u_{xx} - 4u_{xy} + 5u_{yy} + u_x - 3u_y + 6u + 2x = 0\,;$
2. $u_{xx} - 6u_{xy} + 9u_{yy} + 7u_x - 3 = 0\,;$
3. $2u_{xx} - 4u_{yy} + 3u_x - 5u_y + x + 1 = 0;$
4. $3u_{xx} + u_{xy} + 3u_y - 5u + y = 0;$
5. $5u_{xx} + 16u_{xy} + 16u_{yy} - u_x + 3u_y + 9u + x + y - 2 = 0;$
6. $u_{xx} - yu_{yy} = 0;$
7. $u_{xx} - xu_{yy} = 0;$
8. $u_{xx}\operatorname{sign} y + 2u_{xy} + u_{yy} = 0;$
9. $xu_{xx} - yu_{yy} = 0;$
10. $yu_{xx} - xu_{yy} = 0;$
11. $x^2u_{xx} - y^2u_{yy} = 0;$
12. $x^2u_{xx} + y^2u_{yy} = 0;$

13. $y^2 u_{xx} + x^2 u_{yy} = 0$;

14. $y^2 u_{xx} - x^2 u_{yy} = 0$;

15. $(1+x^2)^2 u_{xx} + u_{yy} + 2x(1+x^2)u_x = 0$;

16. $(1+x^2)u_{xx} + (1+y^2)u_{yy} + xu_x + yu_y - 2u = 0$;

17. $x^2 u_{xx} + 2xyu_{xy} + y^2 u_{yy} - 2yu_x + ye^{y/x} = 0$;

18. $u_{xx}\sin^2 x - 2yu_{xy}\sin x + y^2 u_{yy} = 0$;

19. $4y^2 u_{xx} - e^{2x} u_{yy} = 0$;

20. $x^2 u_{xx} + 2xyu_{xy} - 3y^2 u_{yy} - 2xu_x + 4yu_y + 16x^4 u = 0$;

21. $y^2 u_{xx} - 2yu_{xy} + u_{yy} = 0$;

22. $xy^2 u_{xx} - 2x^2 yu_{xy} + x^3 u_{yy} - y^2 u_x = 0$;

23. $u_{xx} - 2\sin x\, u_{xy} - \cos^2 x\, u_{yy} - \cos x\, u_y = 0$;

24. $xu_{xx} + 2xu_{xy} + (x-1)u_{yy} = 0$.

Solution of Problem 1. The equation is elliptic in the whole plane since

$$a_{12}^2 - a_{11}a_{22} = 2^2 - 1 \cdot 5 = -1 < 0.$$

The characteristic equation has the form

$$\frac{dy}{dx} = -2 \pm i,$$

and its general integral is

$$y + 2x \pm ix = C.$$

The change of variables

$$\xi = 2x + y, \quad \eta = x$$

yields

$$u_x = 2u_\xi + u_\eta, \quad u_y = u_\xi,$$

$$u_{xx} = 4u_{\xi\xi} + 4u_{\xi\eta} + u_{\eta\eta},$$

$$u_{xy} = 2u_{\xi\xi} + u_{\xi\eta}, \quad u_{yy} = u_{\xi\xi}.$$

Therefore, we reduce the equation to the canonical form

$$u_{\xi\xi} + u_{\eta\eta} - u_\xi + u_\eta + 6u + 2\eta = 0.$$

Solution of Problem 6. One has $a_{12}^2 - a_{11}a_{22} = y$; hence the equation is hyperbolic for $y > 0$, it is elliptic for $y < 0$, and $y = 0$ is the line where the equation is parabolic. In the domain $y > 0$, general integrals of the characteristic equation have the form $x \pm 2\sqrt{y} = C$. The change of variables

$$\xi = x + 2\sqrt{y}, \quad \eta = x - 2\sqrt{y}$$

yields

$$u_x = u_\xi + u_\eta, \quad u_y = (u_\xi - u_\eta)\frac{1}{\sqrt{y}},$$

$$u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta},$$

$$u_{yy} = (u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta})\frac{1}{y} - (u_\xi - u_\eta)\frac{1}{2y\sqrt{y}},$$

$$\xi - \eta = 4\sqrt{y}.$$

Hence, we get the following canonical form:

$$u_{\xi\eta} + \frac{1}{2(\xi - \eta)}(u_\xi - u_\eta) = 0.$$

In the domain $y < 0$, the change of variables

$$\xi = x, \quad \eta = 2\sqrt{-y}$$

yields

$$u_x = u_\xi, \quad u_y = -\frac{1}{\sqrt{-y}}u_\eta = -\frac{2}{\eta}u_\eta,$$

$$u_{xx} = u_{\xi\xi}, \quad u_{yy} = -\frac{1}{y}u_{\eta\eta} + \frac{1}{2y\sqrt{-y}}u_\eta.$$

Hence, we get the following canonical form:

$$u_{\xi\xi} + u_{\eta\eta} - \frac{1}{\eta}u_\eta = 0.$$

6.1.2. Reduce the following equations with constant coefficients to the canonical form, and carry out further simplifications:

1. $2u_{xx} + 2u_{xy} + u_{yy} - 4u_x + u_y + u = 0$;
2. $3u_{xx} - 5u_{xy} - 2u_{yy} + 3u_x + u_y = 2$;
3. $u_{xx} + 2u_{xy} + u_{yy} + u_x + 2u_y + u = 0$;
4. $u_{xx} + u_{xy} - 2u_{yy} + 9u_x - 3u + x = 0$;
5. $u_{xx} + 4u_{xy} + u_{yy} - 3u_y + u = 0$;
6. $4u_{xx} + 4u_{xy} + u_{yy} - 5u_x + 3u_y - 6u = 0$;
7. $u_{xx} - 6u_{xy} + 10u_{yy} + 7u_x - 5u_y + 2u = 0$;
8. $u_{xx} - 2u_{xy} - 3u_{yy} + 2u_x - u_y - 9u = 0$;

9. $u_{xx}+2u_{xy}-3u_{yy}-5u_x+u_y-2u=0;$

10. $u_{xx}+4u_{xy}+5u_{yy}-u_y+3u+2x=0;$

11. $u_{xx}+4u_{xy}+13u_{yy}+3u_x+24u_y-9u+9(x+y)=0;$

12. $u_{xx}-2u_{xy}+u_{yy}+u_x-7u_y-9u=0;$

13. $u_{xx}-u_{yy}+u_x+u_y-4u=0;$

14. $9u_{xx}-6u_{xy}+u_{yy}-8u_x+u_y+x-3y=0;$

15. $u_{xy}+2u_{yy}-u_x+4u_y+u=0;$

16. $u_{xy}+u_{xx}-u_y-10u+4x=0;$

17. $u_{xx}-2u_{xy}+u_{yy}-3u_x+12u_y+27u=0.$

Hint. After the reduction to the canonical form go on to a new unknown function $v(\xi,\eta)$ by the formula $u(\xi,\eta)=e^{\lambda\xi+\mu\eta}v(\xi,\eta)$. Choose the parameters λ and μ such that the terms with first derivatives or with the unknown function are absent.

Solution of Problem 1. The equation is elliptic in the whole plane since

$$a_{12}^2-a_{11}a_{22}=1^2-2\cdot 1=-1<0.$$

The characteristic equation has the form

$$\frac{dy}{dx}=\frac{1}{2}\pm\frac{1}{2}i$$

and its general integral is

$$y-\frac{1}{2}x\pm\frac{1}{2}ix=C.$$

The change of variables

$$\xi=y-\frac{x}{2},\quad \eta=\frac{x}{2}$$

reduces the equation to the canonical form

$$u_{\xi\xi}+u_{\eta\eta}+6u_\xi-4u_\eta+2u=0.$$

Using the replacement $u(\xi,\eta)=e^{\lambda\xi+\mu\eta}v(\xi,\eta)$ we obtain the following equation with respect to $v(\xi,\eta)$:

$$v_{\xi\xi}+v_{\eta\eta}+(2\lambda+6)v_\xi+(2\mu-4)v+(\lambda^2+\mu^2+6\lambda-4\mu+2)v=0.$$

Taking $\lambda=-3$, $\mu=2$, we get

$$v_{\xi\xi}+v_{\eta\eta}-11v=0.$$

6.1.3. Reduce the following equations to the canonical form:

1. $2u_{xx} + u_{yy} + 6u_{zz} + 2u_{xy} - 2u_{yz} = 0;$
2. $5u_{xx} + 2u_{yy} + u_{zz} + 4u_{xy} + 2u_{yz} = 0;$
3. $3u_{yy} - 2u_{xy} - 2u_{yz} + 7u = 0;$
4. $4u_{xx} + u_{yy} + u_{zz} + 4u_{xy} + 4u_{xz} + 2u_{yz} + 5u = 0;$
5. $4u_{xx} - 2u_{yz} - 4u_{xy} + u_y + u_z = 0;$
6. $9u_{xx} + 4u_{yy} + u_{zz} + 12u_{xy} + 6u_{xz} + 4u_{yz} - 6u_x - 4u_y - 2u_z = 0;$
7. $u_{tt} + 3u_{xx} + 2u_{yy} + 2u_{zz} + 2u_{ty} + 2u_{yz} + 2u_{xy} = 0;$
8. $u_{tt} + 2u_{zz} + u_{xy} - 2u_{zt} - u_{xz} = 0;$
9. $u_{xx} + u_{zz} + 2u_{xy} - 2u_{xz} - 4u_{yz} + 2u_{yt} = 0.$

6.1.4. Find the general solutions for the following equations:

1. $2u_{xx} - 5u_{xy} + 3u_{yy} = 0;$
2. $2u_{xx} + 6u_{xy} + 4u_{yy} + u_x + u_y = 0;$
3. $3u_{xx} - 10u_{xy} + 3u_{yy} - 2u_x + 4u_y + \dfrac{5}{16}u = 0;$
4. $3u_{xx} + 10u_{xy} + 3u_{yy} + u_x + u_y + \dfrac{1}{16}u - 16x\exp\left(-\dfrac{x+y}{16}\right) = 0;$
5. $u_{yy} - 2u_{xy} + 2u_x - u_y = 4e^x;$
6. $u_{xx} - 6u_{xy} + 8u_{yy} + u_x - 2u_y + 4\exp\left(5x + \dfrac{3y}{2}\right) = 0;$
7. $u_{xx} - 2\cos x u_{xy} - (3 + \sin^2 x)u_{yy} + u_x + (\sin x - \cos x - 2)u_y = 0;$
8. $e^{-2x}u_{xx} - e^{-2y}u_{yy} - e^{-2x}u_x + e^{-2y}u_y + 8e^y = 0;$
9. $u_{xy} + yu_y - u = 0.$

6.2. Hyperbolic Partial Differential Equations

Hyperbolic partial differential equations usually describe oscillation processes and give a mathematical description of wave propagation. The prototype of the class of hyperbolic equations and one of the most important differential equations of mathematical physics is the wave equation. Hyperbolic equations occur in such diverse fields of study as electromagnetic theory, hydrodynamics, acoustics, elasticity and quantum theory. In order to solve exercises from this section use the theory from Chapter 2.

1. Method of travelling waves

6.2.1. Solve the following problems of oscillations of a homogeneous infinite string $(-\infty < x < \infty)$ if $a = 1$ in (2.1.1), and the initial displacements and the initial velocities have the form:

1. $u(x,0) = \sin x, \quad u_t(x,0) = 0;$
2. $u(x,0) = 0, \quad u_t(x,0) = A\sin x, \quad A - const;$
3. $u(x,t_0) = \varphi(x), \quad u_t(x,t_0) = \psi(x), \quad t > t_0;$
4. $u(x,0) = x^2, \quad u_t(x,0) = 4x.$

6.2.2. Find the solution of the equation

$$u_{xx} + 2u_{xy} - 3u_{yy} = 0, \quad y > 0, \quad -\infty < x < \infty,$$

satisfying the initial conditions

$$u(x,0) = 3x^2, \quad u_y(x,0) = 0.$$

Solution. The equation is hyperbolic in the whole plane since

$$a_{12}^2 - a_{11}a_{22} = 1 + 3 = 4 > 0,$$

The characteristic equations have the form

$$\frac{dy}{dx} = -1, \quad \frac{dy}{dx} = 3.$$

The change of variables

$$\xi = x + y, \quad \eta = 3x - y$$

reduces the equation to the canonical form

$$u_{\xi\eta} = 0.$$

The general solution of this equation is $u = f(\xi) + g(\eta)$, and consequently,

$$u(x,y) = f(x+y) + g(3x-y),$$

where f and g are arbitrary twice continuously differentiable functions. Using the initial conditions we calculate

$$f(x) + g(3x) = 3x^2, \quad f'(x) + g'(3x) = 0,$$

hence

$$f(x) = \frac{3x^2}{4}, \quad g(x) = \frac{x^2}{4}.$$

This yields

$$u(x,y) = 3x^2 + y^2.$$

6.2.3. Find the solution of the equation

$$4y^2 u_{xx} + 2(1-y^2)u_{xy} - u_{yy} - \frac{2y}{1+y^2}(2u_x - u_y) = 0,$$

satisfying the initial conditions

$$u(x,0) = \varphi_0(x), \quad u_y(x,0) = \varphi_1(x).$$

6.2.4. Find the solution of the equation

$$u_{xx} + 2\cos x\, u_{xy} - \sin^2 x\, u_{yy} - \sin x\, u_y = 0, \quad y > 0,\ -\infty < x < \infty,$$

satisfying the initial conditions

$$u_{|y=\sin x} = \varphi_0(x), \quad u_{y|y=\sin x} = \varphi_1(x).$$

6.2.5. Find the solution of the equation

$$u_{xx} + y u_{yy} + \frac{1}{2} u_y = 0, \quad y > 0, \quad -\infty < x < \infty,$$

satisfying the conditions

$$u(x,0) = f(x), \quad u_y(x,0) = 0.$$

6.2.6. Solve the following Cauchy problems in the domain $t > 0,\ -\infty < x < \infty$:

1. $u_{tt} = u_{xx} + 6, \quad u(x,0) = x^2, \quad u_t(x,0) = 4x;$
2. $u_{tt} = u_{xx} + xt, \quad u(x,0) = x, \quad u_t(x,0) = \sin x;$
3. $u_{tt} = u_{xx} + \sin x, \quad u(x,0) = \sin x, \quad u_t(x,0) = 0;$
4. $u_{tt} = u_{xx} + \sin x, \quad u(x,0) = 0, \quad u_t(x,0) = 0;$
5. $u_{tt} = u_{xx} + x\sin t, \quad u(x,0) = \sin x, \quad u_t(x,0) = \cos x;$
6. $u_{tt} = u_{xx} + e^x, \quad u(x,0) = \sin x, \quad u_t(x,0) = x + \cos x;$
7. $u_{tt} = u_{xx} + e^{-t}, \quad u(x,0) = \sin x, \quad u_t(x,0) = \cos x.$

6.2.7. Using the reflection method (see Section 2.1) solve the following problems on the half-line:

1. $u_{tt} = a^2 u_{xx},\ u_x(0,t) = 0,\ u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x);$
2. $u_{tt} = a^2 u_{xx},\ u_x(0,t) = u(x,0) = 0,\ u_t(x,0) = \begin{cases} 0, & 0 < x < l, \\ v_0, & l < x < 2l, \\ 0, & 2l < x < \infty; \end{cases}$

3. $u_{tt}=a^2u_{xx},\ u(0,t)=A\sin\omega t,\ u(x,0)=0,\ u_t(x,0)=0.$

6.2.8. A semi-infinite homogeneous string ($x\ge 0$) with the fixed end-point $x=0$ is perturbed by the initial displacement

$$u(x,0)=\begin{cases} 0, & 0\le x\le e,\\ -\sin\dfrac{\pi x}{e}, & e\le x\le 2e,\\ 0, & 2e\le x<\infty. \end{cases}$$

Find the form of the string at

$$t=\frac{e}{4a},\ \frac{e}{a},\ \frac{5e}{4a},\ \frac{3e}{a},\ \frac{7e}{a},$$

provided that the initial velocity is equal to zero, i.e. $u_t(x,0)=0$.

6.2.9. A homogeneous string of length l ($0\le x\le l$) is fixed at the ends $x=0$ and $x=l$. At initial time $t=0$ the string is deviated at the point $x=l/3$ to the distance h from the axis Ox, and then is released without initial velocity. Show that for $0\le t\le\frac{1}{3a}$ the form of the string is expressed by the formula

$$u(x,t)=\begin{cases} \dfrac{3hx}{l}, & \text{for } 0<x\le\dfrac{l}{3}-at,\\ \dfrac{3h}{4l}x+\dfrac{9h}{4l}(\dfrac{l}{3}-at), & \text{for } \dfrac{l}{3}-at<x\le\dfrac{l}{3}+at,\\ \dfrac{3h}{2l}(l-x), & \text{for } \dfrac{l}{3}+at<x<l. \end{cases}$$

6.2.10. Solve the following problems in the domain $0\le x\le l,\ t>a$:

1. $u_{xx}=\dfrac{1}{a^2}u_{tt},$

   $u(0,t)=0,\quad u(l,t)=0,\quad t>0,$

   $u(x,0)=A\sin\dfrac{\pi x}{l},\quad u_t(x,0)=0,\quad 0\le x\le l\,;$

2. $u_{xx}=\dfrac{1}{a^2}u_{tt},$

   $u(0,t)=0,\quad u_x(l,t)=0,\quad t>0,$

   $u(x,0)=Ax,\quad u_t(x,0)=0,\quad 0\le x\le l.$

Solution of Problem 1. Consider the following Cauchy problem

$$u_{xx}=\frac{1}{a^2}u_{tt},\quad -\infty<x<\infty,\quad t>0,$$

$$u(x,0)=A\sin\frac{\pi x}{l},\quad u_t(x,0)=0.$$

According to the d'Alambert's formula (2.1.4), the solution of this Cauchy problem has the form

$$u(x,t) = A\sin\frac{\pi x}{l}\cos\frac{a\pi t}{l}.$$

It is easy to check that this function also is the solution of problem 6.2.10:1.

6.2.11. Solve the following problem in the domain $-\infty < x < \infty,\ t \geq 0$:

$$u_{tt} = a^2 u_{xx},\ u(x,0) = f(x),\ u_t(x,0) = -af'(x),$$

where f is a smooth function (see fig. 6.2.1).

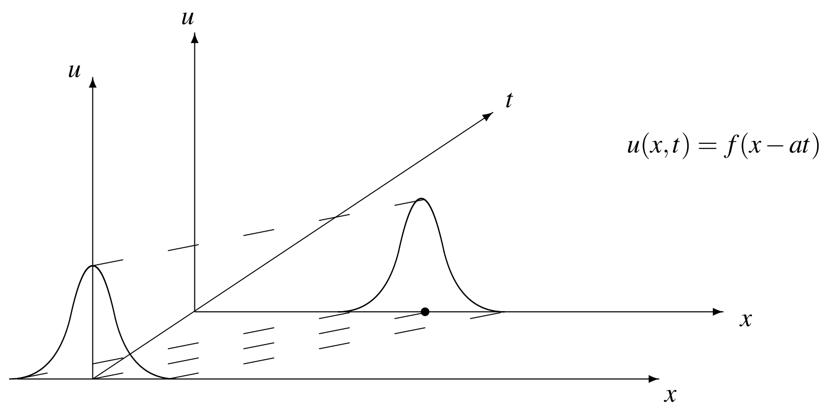

Figure 6.2.1.

2. Method of separation of variables

Sturm-Liouville problem. Consider the following boundary value problem L

$$-(k(x)y'(x))' + q(x)y(x) = \lambda\rho(x)y(x), \quad 0 \leq x \leq l,$$

$$h_1 y'(0) + hy(0) = H_1 y'(l) + Hy(l) = 0,$$

which is called the Sturm-Liouville problem. Here λ is the spectral parameter, k, q, ρ are real-valued functions, and h_1, h, H_1, H are real numbers. We assume that $\rho(x), k(x) \in W_2^1[0,l]$ (i.e. $\rho(x), k(x), \rho'(x), k'(x)$ are absolutely continuous functions), $q(x) \in L(0,l)$, $\rho(x) > 0, k(x) > 0$, and $|h_1| + |h| > 0$, $|H_1| + |H| > 0$. We are interested in non-trivial solutions of the boundary value problem L.

The values of the parameter λ for which L has nonzero solutions are called *eigenvalues,* and the corresponding nontrivial solutions are called *eigenfunctions.*

It is known that L has a countable set of eigenvalues $\{\lambda_n\}_{n\geq 0}$, and $\lambda_n = O(n^2)$ as $n \to \infty$. For each of these eigenvalues λ_n there exists only one eigenfunction (up to a multiplicative constant) $y_n(x)$. The set of eigenfunctions $\{y_n(x)\}_{n\ge 0}$ form a complete and orthonormal system in $L_2[0,l]$ with the weight $r(x)$, i.e.:

$$\int_0^l \rho(x)y_n(x)y_m(x)\,dx = \begin{cases} 0, & n\neq m, \\ 1, & n=m, \end{cases}$$

If $f(x)$ is a twice continuously differentiable function satisfying boundary conditions, then $f(x)$ can be expanded into the uniformly convergent series:

$$f(x)=\sum_{n=1}^{\infty} a_n y_n(x),$$

where

$$a_n=\int_0^l \rho(x)f(x)y_n(x)\,dx.$$

Note that for the case $h_1H_1\neq 0$, these facts were proved in section 2.2 (see Theorem 2.2.8). Other cases can be treated similarly.

6.2.11. Find eigenvalues and eigenfunctions of the boundary value problems for the equation $y''+\lambda y=0$ ($0<x<l$) with the following boundary conditions:

1. $y(0)=y(l)=0;$
2. $y'(0)=y'(l)=0;$
3. $y(0)=y'(l)=0;$
4. $y'(0)=y(l)=0;$
5. $y'(0)=y'(l)+hy(l)=0.$

Solution of Problem 1. Let $\lambda=\rho^2$. The general solution of the equation $y''+\lambda y=0$ has the form

$$y(x)=C_1\frac{\sin\rho x}{\rho}+C_2\cos\rho x.$$

Using the boundary condition $y(0)=0$, we get $C_2=0$. The second boundary condition $y(l)=0$ gives us the equation that must be satisfied by the eigenvalues:

$$\frac{\sin\rho l}{\rho}=0.$$

Hence, the eigenvalues and the eigenfunctions of Problem 1 have the form

$$\lambda_n=\left(\frac{n\pi}{l}\right)^2, \quad y_n(x)=\sin\frac{n\pi}{l}x, \quad n=1,2,\ldots$$

6.2.12. Find eigenvalues and eigenfunctions of the following boundary value problems $(0<x<l)$:

1. $-y''+\gamma y=\lambda y, \quad y(0)=y(l)=0;$

2. $-y'' + \gamma y = \lambda y, \quad y'(0) = y'(l) = 0;$

3. $-y'' + \gamma y = \lambda y, \quad y(0) = y'(l) = 0;$

4. $-y'' + \eta y' + \gamma y = \lambda y, \quad y(0) = y(l) = 0.$

Solution of Problem 4. The replacement

$$y(x) = \exp\left(\frac{\eta x}{2}\right) Y(x)$$

yields

$$y'(x) = \exp\left(\frac{\eta x}{2}\right)\left(Y'(x) + \frac{\eta}{2} Y(x)\right),$$

$$y''(x) = \exp\left(\frac{\eta x}{2}\right)\left(Y''(x) + \eta Y'(x) + \frac{\eta^2}{4} Y(x)\right),$$

and consequently,

$$Y''(x) + \mu Y(x) = 0, \quad Y(0) = Y(l) = 0,$$

where $\mu = \lambda - \frac{\eta^2}{4} - \gamma$. The eigenvalues and the eigenfunctions of this boundary value problem are

$$\mu_n = \left(\frac{n\pi}{l}\right)^2, \quad Y_n(x) = \sin\frac{n\pi}{l}x, \quad n \geq 1$$

(see Problem 6.2.11). Therefore, the eigenvalues and the eigenfunctions of Problem 4 have the form

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2 + \frac{\eta^2}{4} + \gamma, \quad y_n(x) = \exp\left(\frac{\eta x}{2}\right)\sin\frac{n\pi}{l}x, \quad n \geq 1.$$

Let us briefly present the scheme of the method of separation of variables for solving the mixed problem for a vibrating string. Consider the following mixed problem for the equation

$$\rho(x)u_{tt} = (k(x)u_x)_x - q(x)u, \quad 0 \leq x \leq l,\ t \geq 0 \tag{6.2.1}$$

with the initial conditions

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x) \tag{6.2.2}$$

and with the boundary conditions

$$(h_1 u_x + hu)|_{x=0} = (H_1 u_x + Hu)|_{x=l} = 0. \tag{6.2.3}$$

First we consider the following *auxiliary problem*. We will seek nontrivial (i.e. not identically zero) particular solutions of equation (6.2.1) such that they satisfy the boundary conditions (6.2.3) and admit separation of variables, i.e. they have the form

$$u(x,t) = T(t)y(x). \tag{6.2.4}$$

Substituting (6.2.4) into (6.2.1) and (6.2.3), we get the Sturm-Liouville problem L for the function $y(x)$, and the equation $T''(t) + \lambda T(t) = 0$ for the function $T(t)$.

Let $\lambda_n, y_n(x)$ $(n = 0, 1, 2, \ldots)$ be the eigenvalues and the eigenfunctions of the problem L. Then the functions

$$u_n(x,t) = \left(A_n \cos\sqrt{\lambda_n}t + B_n \frac{\sin\sqrt{\lambda_n}t}{\sqrt{\lambda_n}}\right) y_n(x) \tag{6.2.5}$$

satisfy the equation (6.2.1) and the boundary conditions (6.2.3) for any A_n and B_n. The solutions of the form (6.2.5) are called *standing waves* (normal modes of vibrations).

We will seek the solution of the mixed problem (6.2.1)-(6.2.3) by superposition of standing waves (6.2.5):

$$u(x,t) = \sum_{n=0}^{\infty} \left(A_n \cos\sqrt{\lambda_n}t + B_n \frac{\sin\sqrt{\lambda_n}t}{\sqrt{\lambda_n}}\right) y_n(x). \tag{6.2.6}$$

Next we use the initial conditions (6.2.2) for finding the coefficients A_n and B_n. For this purpose we substitute (6.2.6) into (6.2.2) and calculate

$$\varphi(x) = \sum_{n=0}^{\infty} A_n y_n(x), \quad \psi(x) = \sum_{n=0}^{\infty} B_n y_n(x).$$

Then the coefficients A_n and B_n are given by the formulae

$$A_n = \int_0^l \rho(x)\varphi(x)y_n(x)\,dx, \quad B_n = \int_0^l \rho(x)\psi(x)y_n(x)\,dx.$$

6.2.13. Solve the following problems of oscillations of a homogeneous string $(0 < x < l)$ with fixed end-points if the initial displacement $u(x,0)$ and the initial velocities $u_t(x,0)$ have the form:

1. $u(x,0) = A\sin\dfrac{\pi x}{l}, \quad u_t(x,0) = 0;$

2. $u(x,0) = \dfrac{h}{l^2}x(l-x), \quad u_t(x,0) = 0;$

3. $u(x,0) = \dfrac{hx}{l}$ $(0 \le x \le C)$, $u(x,0) = \dfrac{h(l-x)}{l-C}$ $(C \le x \le l)$,
   $u_t(x,0) = 0;$

4. $u(x,0) = 0, \quad u_t(x,0) = v_0 - const;$

5. $u(x,0) = 0,$
   $u_t(x,0) = v_0$ $(\alpha \le x \le \beta)$, $\quad u_t(x,0) = 0$ $(x \notin [\alpha,\beta])$;

6. $u(x,0) = 0, \quad u_t(x,0) = \sin\dfrac{2\pi}{l}x.$

Solution of Problem 1. We have to solve the following mixed problem

$$u_{tt} = a^2 u_{xx}, \quad 0 < x < l,\ t > 0,$$

$$u_{|x=0} = u_{|x=l} = 0,$$
$$u_{|t=0} = A \sin \frac{\pi}{l} x, \quad u_{t|t=0} = 0.$$

This problem is a particular case of the mixed problem (2.2.1)-(2.2.3) when $\varphi(x) = A \sin \frac{\pi}{l} x$, $\psi(x) = 0$. According to (2.2.17)-(2.2.18) the solution has the form

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \sin \frac{an\pi}{l} t + B_n \cos \frac{an\pi}{l} t \right) \sin \frac{n\pi}{l} x,$$

where

$$A_n = 0, \quad B_n = \frac{2}{l} \int_0^l A \sin \frac{\pi}{l} x \sin \frac{n\pi}{l} x \, dx.$$

Clearly, $B_1 = A$ and $B_n = 0$ for $n \geq 2$, and consequently,

$$u(x,t) = A \cos \frac{a\pi}{l} t \sin \frac{\pi}{l} x.$$

6.2.14. Solve the following problems of oscillations of a homogeneous string $(0 < x < l)$ with free end-points if the initial displacement $u(x,0)$ and the initial velocities $u_t(x,0)$ have the form:

1. $u(x,0) = 1, \quad u_t(x,0) = 0;$
2. $u(x,0) = x, \quad u_t(x,0) = 1;$
3. $u(x,0) = \cos \frac{\pi x}{l}, \quad u_t(x,0) = 0;$
4. $u(x,0) = 0, \quad u_t(x,0) = v_0 \ (\alpha \leq x \leq \beta), \quad u_t(x,0) = 0 \ (x \notin [\alpha, \beta]).$

Solution of Problem 2. We have to solve the following mixed problem

$$u_{tt} = a^2 u_{xx}, \quad 0 < x < l,\ t > 0,$$

$$u_{x|x=0} = u_{x|x=l} = 0,$$
$$u_{|t=0} = x, \quad u_{t|t=0} = 1.$$

First we seek nontrivial particular solutions of the equation of the form $u(x,t) = Y(x)T(t)$, which satisfy the boundary conditions. Repeating the arguments from section 2.2 we obtain for the function $T(t)$ the ordinary differential equation

$$\ddot{T}(t) + a^2 \lambda T(t) = 0,$$

and for the function $Y(x)$ we get the Sturm-Liouville boundary value problem

$$Y''(x)+\lambda Y(x)=0, \quad Y'(0)=Y'(l)=0.$$

Here λ is the spectral parameter. The eigenvalues and the eigenfunctions of this boundary value problem are

$$\lambda_n=\Big(\frac{n\pi}{l}\Big)^2, \quad Y_n(x)=\cos\frac{n\pi}{l}x, \quad n\ge 0.$$

The corresponding equations

$$\ddot{T}_n(t)+a^2\lambda_n T_n(t)=0, \quad ,n\ge 0,$$

have the general solutions

$$T_0(t)=A_0t+B_0, \quad T_n(t)=A_n\sin\frac{an\pi}{l}t+B_n\cos\frac{an\pi}{l}t,\ n\ge 1,$$

where A_n and B_n are arbitrary constants. Hence,

$$u_0(x,t)=A_0t+B_0,$$

$$u_n(x,t)=\Big(A_n\sin\frac{an\pi}{l}t+B_n\cos\frac{an\pi}{l}t\Big)\cos\frac{n\pi}{l}x, \quad n\ge 1.$$

We seek the solution of the mixed problem of the form

$$u(x,t)=A_0t+B_0+\sum_{n=1}^{\infty}\Big(A_n\sin\frac{an\pi}{l}t+B_n\cos\frac{an\pi}{l}t\Big)\cos\frac{n\pi}{l}x,$$

and we find A_n and B_n from the initial conditions:

$$x=B_0+\sum_{n=1}^{\infty}B_n\cos\frac{n\pi}{l}x,$$

$$1=A_0+\sum_{n=1}^{\infty}\frac{an\pi}{l}A_n\cos\frac{n\pi}{l}x,$$

and hence

$$B_0=\frac{1}{l}\int_0^l x\,dx=\frac{l}{2}, \quad A_0=\frac{1}{l}\int_0^l dx=1,$$

$$B_n=\frac{2}{l}\int_0^l x\cos\frac{n\pi}{l}x\,dx=\frac{2l}{n^2\pi^2}\left((-1)^n-1\right), \quad n\ge 1,$$

$$A_n=\frac{2}{an\pi}\int_0^l\cos\frac{n\pi}{l}x\,dx=0, \quad n\ge 1.$$

This yields

$$u(x,t)=t+\frac{l}{2}+\sum_{n=1}^{\infty}\frac{2l}{n^2\pi^2}\left((-1)^n-1\right)\cos\frac{an\pi}{l}t\cos\frac{n\pi}{l}x.$$

6.2.15. Solve the following problems of oscillations of a homogeneous string $(0<x<l)$ if the initial and boundary conditions have the form:

1. $u(x,0) = x,\ u_t(x,0) = \sin\frac{\pi}{2l}x,\ u(0,t) = u_x(l,t) = 0;$

2. $u(x,0) = \cos\frac{\pi}{2l}x,\ u_t(x,0) = 0,\ u_x(0,t) = u(l,t) = 0;$

3. $u(x,0) = 0,\ u_t(x,0) = 1,\ u_x(0,t) = u_x(l,t) + hu(l,t) = 0;$

4. $u(x,0) = \sin\frac{3\pi}{2l}x,\ u_t(x,0) = \cos\frac{\pi}{2l}x,\ u(0,t) = u_x(l,t) = 0.$

Solution of Problem 2. We have to solve the following mixed problem

$$u_{tt} = a^2 u_{xx}, \quad 0 < x < l,\ t > 0,$$

$$u_{x|x=0} = u_{|x=l} = 0,$$
$$u_{|t=0} = \cos\frac{\pi}{2l}x, \quad u_{t|t=0} = 0.$$

First we seek nontrivial particular solutions of the equation of the form $u(x,t) = Y(x)T(t)$, which satisfy the boundary conditions. Repeating the arguments from section 2.2 we obtain for the function $T(t)$ the ordinary differential equation

$$\ddot{T}(t) + a^2\lambda T(t) = 0,$$

and for the function $Y(x)$ we get the Sturm-Liouville boundary value problem

$$Y''(x) + \lambda Y(x) = 0, \quad Y'(0) = Y(l) = 0.$$

The eigenvalues and the eigenfunctions of this boundary value problem are

$$\lambda_n = \left(\frac{(2n+1)\pi}{2l}\right)^2, \quad Y_n(x) = \cos\frac{(2n+1)\pi}{2l}x, \quad n \geq 0.$$

The corresponding equations

$$\ddot{T}_n(t) + a^2\lambda_n T_n(t) = 0, \quad ,n \geq 0,$$

have the general solutions

$$T_n(t) = A_n \sin\frac{a(2n+1)\pi}{2l}t + B_n \cos\frac{a(2n+1)\pi}{2l}t,\ n \geq 0,$$

where A_n and B_n are arbitrary constants. Hence,

$$u_n(x,t) = \left(A_n \sin\frac{a(2n+1)\pi}{2l}t + B_n \cos\frac{a(2n+1)\pi}{2l}t\right)\cos\frac{(2n+1)\pi}{2l}x.$$

We seek the solution of the mixed problem of the form

$$u(x,t) = \sum_{n=0}^{\infty}\left(A_n \sin\frac{a(2n+1)\pi}{2l}t + B_n \cos\frac{a(2n+1)\pi}{2l}t\right)\cos\frac{(2n+1)\pi}{2l}x,$$

and we find A_n and B_n from the initial conditions:

$$\cos\frac{\pi}{2l}x = \sum_{n=0}^{\infty} B_n \cos\frac{(2n+1)\pi}{2l}x,$$

$$0 = \sum_{n=0}^{\infty} \frac{a(2n+1)\pi}{2l} A_n \cos\frac{(2n+1)\pi}{2l}x,$$

and hence

$$B_0 = 1, \quad B_n = 0,\ n \ge 1, \qquad A_n = 0, \quad n \ge 0.$$

This yields

$$u(x,t) = \cos\frac{a\pi}{2l}t\cos\frac{\pi}{2l}x.$$

6.2.16. Solve the following mixed problems applying the method of separation of variables:

1. $u_{tt} = u_{xx} - 4u, \quad 0 < x < 1,$
   $u(x,0) = x^2 - x, \quad u_t(x,0) = 0, \quad u(0,t) = u(1,t) = 0;$
2. $u_{tt} = u_{xx} + u, \quad 0 < x < \pi,$
   $u(x,0) = 0, \quad u_t(x,0) = 1, \quad u_x(0,t) = u_x(\pi,t) = 0;$
3. $u_{tt} = t^{\alpha} u_{xx}, \quad (\alpha > -1), \quad 0 < x < l,$
   $u(x,0) = \varphi(x), \quad u_t(x,0) = 0, \quad u(0,t) = u(l,t) = 0;$
4. $u_{tt} + 2u_t = u_{xx} - u, \quad 0 < x < \pi,$
   $u(x,0) = \pi x^2 - x^2, \quad u_t(x,0) = 0, \quad u(0,t) = u(\pi,t) = 0;$
5. $u_{tt} + 2u_t = u_{xx} - u, \quad 0 < x < \pi,$
   $u(x,0) = 0, \quad u_t(x,0) = x, \quad u_x(0,t) = u(\pi,t) = 0;$
6. $u_{tt} = a^2 (xu_x)_x, \quad 0 < x < l,$
   $u(x,0) = \varphi(x), \quad u_t(x,0) = 0, \quad |u(0,t)| < \infty, \quad u(l,t) = 0.$

Solution of Problem 1. First we seek nontrivial particular solutions of the equation of the form $u(x,t) = Y(x)T(t)$, which satisfy the boundary conditions. We obtain for the function $T(t)$ the differential equation

$$\ddot{T}(t) + \lambda T(t) = 0,$$

and for the function $Y(x)$ we get the Sturm-Liouville boundary value problem

$$Y''(x) + (\lambda - 4)Y(x) = 0, \quad Y(0) = Y(l) = 0.$$

The eigenvalues and the eigenfunctions of this boundary value problem are

$$\lambda_n = (n\pi)^2 + 4, \quad Y_n(x) = \sin n\pi x, \quad n \ge 1.$$

The corresponding equations

$$\ddot{T}_n(t) + \lambda_n T_n(t) = 0, \quad n \geq 1,$$

have the general solutions

$$T_n(t) = A_n \cos \rho_n t + B_n \sin \frac{\rho_n t}{\rho_n}, \quad n \geq 1,$$

where A_n and B_n are arbitrary constants, and $\rho_n = \sqrt{(n\pi)^2 + 4}$. Hence,

$$u_n(x,t) = \left(A_n \cos \rho_n t + B_n \sin \frac{\rho_n t}{\rho_n} \right) \sin n\pi x, \quad n \geq 1.$$

We seek the solution of the mixed problem of the form

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos \rho_n t + B_n \sin \frac{\rho_n t}{\rho_n} \right) \sin n\pi x,$$

and we find A_n and B_n from the initial conditions:

$$x^2 - x = \sum_{n=1}^{\infty} A_n \sin n\pi x, \quad 0 = \sum_{n=1}^{\infty} B_n \sin n\pi x,$$

and hence

$$A_n = 2 \int_0^1 (x^2 - x) \sin n\pi x \, dx = \frac{4}{n^3\pi^3} \left((-1)^n - 1 \right), \quad B_n = 0, \quad n \geq 1.$$

This yields

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4}{n^3\pi^3} \left((-1)^n - 1 \right) \cos(\sqrt{n^2\pi^2 + 4t}) \sin n\pi x.$$

Solution of Problem 5. First we seek nontrivial particular solutions of the equation of the form $u(x,t) = Y(x)T(t)$, which satisfy the boundary conditions. We obtain for the function $T(t)$ the differential equation

$$\ddot{T}(t) + 2\dot{T}(t) + \lambda T(t) = 0,$$

and for the function $Y(x)$ we get the Sturm-Liouville boundary value problem

$$Y''(x) + (\lambda - 1)Y(x) = 0, \quad Y'(0) = Y(\pi) = 0.$$

The eigenvalues and the eigenfunctions of this boundary value problem are

$$\lambda_n = \left(n + \frac{1}{2} \right)^2 + 1, \quad Y_n(x) = \cos \left(n + \frac{1}{2} \right) x, \quad n \geq 0.$$

The corresponding equations

$$\ddot{T}_n(t) + \dot{T}_n(t) + \lambda_n T_n(t) = 0, \quad n \geq 1,$$

have the general solutions

$$T_n(t) = e^{-t}\left(A_n \cos\left(n+\frac{1}{2}\right)t + B_n \sin\left(n+\frac{1}{2}\right)t\right),\ n \ge 0,$$

where A_n and B_n are arbitrary constants. Hence,

$$u_n(x,t) = e^{-t}\left(A_n \cos\left(n+\frac{1}{2}\right)t + B_n \sin\left(n+\frac{1}{2}\right)t\right)\cos\left(n+\frac{1}{2}\right)x.$$

We seek the solution of the mixed problem of the form

$$u(x,t) = \sum_{n=0}^{\infty} e^{-t}\left(A_n \cos\left(n+\frac{1}{2}\right)t + B_n \sin\left(n+\frac{1}{2}\right)t\right)\cos\left(n+\frac{1}{2}\right)x,$$

and we find A_n and B_n from the initial conditions:

$$A_n = 0, \quad B_n = \frac{8(-1)^n}{(2n+1)^2} - \frac{16}{\pi(2n+1)^3}.$$

This yields

$$u(x,t) = e^{-t}\sum_{n=0}^{\infty} B_n \sin\left(n+\frac{1}{2}\right)t \cos\left(n+\frac{1}{2}\right)x.$$

The method of separation of variables allows us to construct solutions of mixed problems for non-homogeneous equations with boundary conditions.

Consider the problem of forced oscillations of a non-homogeneous string under the influence of an exterior force of density f:

$$\rho(x)u_{tt} = (k(x)u_x)_x - q(x)u + f(x,t), \tag{6.2.7}$$

$$\rho(x) > 0,\ k(x) > 0, \quad 0 < x < l,\ t > 0,$$

$$u|_{t=0} = \varphi(x), \quad u_t|_{t=0} = \psi(x), \tag{6.2.8}$$

$$(h_1 u_x + hu)|_{x=0} = \mu(t), \quad (H_1 u_x + Hu)|_{x=l} = \nu(t). \tag{6.2.9}$$

In the case of homogeneous boundary conditions $(\mu(t) \equiv \nu(t) \equiv 0)$ we seek the solution of problem (6.2.7)-(6.2.9) in the form of a series

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) y_n(x), \tag{6.2.10}$$

with respect to the eigenfunctions of problem L. Substituting (6.2.10) into (6.2.7) and (6.2.8) we obtain the following Cauchy problem for $T_n(t)$:

$$\ddot{T}_n(t) + \lambda_n T_n(t) = f_n(t),$$

$$T_n(0) = \varphi_n,\ T_n'(0) = \psi_n,$$

where $f_n(t), \varphi_n, \psi_n$ are the Fourier coefficients for the functions

$$\frac{1}{\rho(x)} f(x,t), \quad \varphi(x), \quad \psi(x),$$

respectively:

$$f_n(t) = \int_0^l f(x,t) y_n(x)\, dx,$$

$$\varphi_n = \int_0^l \rho(x)\varphi(x) y_n(x)\, dx,$$

$$\psi_n = \int_0^l \rho(x)\psi(x) y_n(x)\, dx.$$

Solving the Cauchy problem we arrive at

$$T_n(t) = \varphi_n \cos\sqrt{\lambda_n} t + \psi_n \frac{\sin\sqrt{\lambda_n} t}{\sqrt{\lambda_n}} + \int_0^t \frac{\sin\sqrt{\lambda_n}(t-\tau)}{\sqrt{\lambda_n}} f_n(\tau)\, d\tau.$$

Problem (6.2.7)-(6.2.9) with non-homogeneous boundary conditions can be reduced to a problem with homogeneous boundary conditions using of the following replacement of the unknown function $u(x,t) = V(x,t) + W(x,t)$, where

$$W(x,t) = (a_{12}x^2 + a_{11}x + a_{10})\mu(t) + (a_{22}x^2 + a_{21}x + a_{20})\nu(t),$$

and where the coefficients a_{ij} are chosen such that the function $W(x,t)$ satisfies the boundary conditions (6.2.9). For example, for the boundary conditions $u(0,t) = \mu(t), u(l,t) = \nu(t)$ one can take

$$W(x,t) = \left(1 - \frac{x}{l}\right)\mu(t) + \frac{x}{l}\nu(t).$$

6.2.17. Solve the problem of forced oscillations of a homogeneous string $(0 < x < l)$ with fixed end-points under the influence of the exterior force of density $f(x,t) = A\sin\omega t$ and with zero initial conditions. Study the resonance effect and find the solution in the case of resonance.

6.2.18. Solve the problem of forced oscillations of a homogeneous string $(0 < x < l)$ under the influence of the exterior force of density $f(x,t)$ and with zero initial conditions provided that:

1. $f(x,t) = f_0, \quad u(0,t) = u(l,t) = 0;$
2. $f(x,t) = f_0, \quad u_x(0,t) = u_x(l,t) = 0;$
3. $f(x,t) = \cos\frac{\pi t}{l}, \quad u(0,t) = u(l,t) = 0;$
4. $f(x,t) = Axe^{-t}, \quad u(0,t) = u(l,t) = 0;$
5. $f(x,t) = Ae^{-t}\sin\frac{\pi}{l}x, \quad u(0,t) = u(l,t) = 0;$

6. $f(x,t) = x, \quad u(0,t) = u_x(l,t) = 0;$

7. $f(x,t) = x(l-x), \quad u(0,t) = u(l,t) = 0.$

Solution of Problem 1. We have to solve the following mixed problem

$$u_{tt} = a^2 u_{xx} + f_0, \quad 0 < x < l,\ t > 0,$$

$$u_{|x=0} = u_{|x=l} = 0,$$

$$u_{|t=0} = u_{t|t=0} = 0.$$

We seek the solution of the form

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin\frac{n\pi}{l}x.$$

We have

$$f_0 = \sum_{n=1}^{\infty} F_n \sin\frac{n\pi}{l}x,$$

where

$$F_n = \frac{2f_0}{l}\int_0^l \sin\frac{n\pi}{l}x\,dx = \frac{2f_0}{n\pi}\left(1-(-1)^n\right).$$

The functions $u_n(t)$ are solutions of the Cauchy problems

$$\ddot{u}_n(t) + \left(\frac{an\pi}{l}\right)^2 u_n(t) = F_n\,,$$

$$u_n(0) = \dot{u}_n(0) = 0.$$

Hence

$$u_n(t) = \frac{F_n l}{an\pi}\int_0^t \sin\frac{an\pi}{l}(t-\tau)\,d\tau$$

$$= \frac{F_n l^2}{(an\pi)^2}\left(1-\cos\frac{an\pi}{l}t\right),$$

and consequently,

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2f_0 l^2}{a^2 n^3 \pi^3}\left(1-(-1)^n\right)\left(1-\cos\frac{an\pi}{l}t\right)\sin\frac{n\pi}{l}x.$$

6.2.19. Solve the problem of forced transverse oscillations of a homogeneous string $(0 < x < l)$ fixed at the end-point $x = 0$ under the influence of the force of density $A\sin\omega t$ applied to the end-point $x = l$ provided that the initial displacement and velocity are equal to zero.

6.2.20. Solve the problem of forced oscillations of a homogeneous string $(0 < x < l)$ under the influence of the exterior force of density $f(x,t)$ provided that:

1. $u(x,0) = u_t(x,0) = 0,\ u(0,t) = \mu_0,\ u(l,t) = \nu_0,\ f(x,t) = 0;$

2. $u(x,0) = u_t(x,0) = 0,\ u_x(0,t) = \mu_0,\ u_x(l,t) = \nu_0,\ f(x,t) = 0;$

3. $u(x,0) = u_t(x,0) = 0,\ u(0,t) = 0,\ u(l,t) = t,\ f(x,t) = 0;$

4. $u(x,0) = 0,\ u_t(x,0) = 1,\ u_x(0,t) = \sin 2t,\ u_x(l,t) = 0,$
   $f(x,t) = \sin 2t;$

5. $u(x,0) = \sin\dfrac{\pi x}{l},\ u_t(x,0) = 0,\ u(0,t) = t^2,\ u(l,t) = t^3,\ f(x,t) = 0;$

6. $u(x,0) = 0,\ u_t(x,0) = 1,\ u(0,t) = t,\ u_x(l,t) = 1,\ f(x,t) = f_0.$

Solution of Problem 1. We have to solve the following mixed problem

$$u_{tt} = a^2 u_{xx}, \quad 0 < x < l,\ t > 0,$$

$$u_{|x=0} = \mu_0\,, \quad u_{|x=l} = \nu_0,$$
$$u_{|t=0} = u_{t|t=0} = 0.$$

Denote

$$w(x) = \left(1 - \frac{x}{l}\right)\mu_0 + \frac{x}{l}\nu_0\,.$$

The replacement $u(x,t) = v(x,t) + w(x)$ yields the following mixed problem with respect to $v(x,t)$:

$$v_{tt} = a^2 v_{xx}, \quad 0 < x < l,\ t > 0,$$
$$v_{|x=0} = 0, \quad v_{|x=l} = 0,$$
$$v_{|t=0} = -w(x), \quad v_{t|t=0} = 0.$$

Applying the method of separation of variables we find

$$v(x,t) = \sum_{n=1}^{\infty} B_n \cos\frac{an\pi}{l}t \sin\frac{n\pi}{l}x\,,$$

where

$$B_n = -\frac{2}{l}\int_0^l w(x)\sin\frac{n\pi}{l}x\,dx = \frac{2}{n\pi}\left(\nu_0(-1)^n - \mu_0\right).$$

Hence

$$u(x,t) = \left(1 - \frac{x}{l}\right)\mu_0 + \frac{x}{l}\nu_0 + \sum_{n=1}^{\infty}\frac{2}{n\pi}\left(\nu_0(-1)^n - \mu_0\right)\cos\frac{an\pi}{l}t \sin\frac{n\pi}{l}x\,.$$

6.2.21. Solve the problem of free oscillations of a homogeneous string $(0 < x < l)$ with fixed end-points in a medium whose resistance is proportional to the first degree of the velocity.

6.2.22. Solve the following problems:

1. $u_{tt} = u_{xx}, \quad 0 < x < 1,$
   $u(x,0) = x + 1, \quad u_t(x,0) = 0, \quad u(0,t) = t + 1, \quad u(1,t) = t^3 + 2;$

2. $u_{tt} = u_{xx} + u, \quad 0 < x < 2,$

   $u(x,0) = 0, \quad u_t(x,0) = 0, \quad u(0,t) = 2t, \quad u(2,t) = 0;$

3. $u_{tt} + u_t = u_{xx}, \quad 0 < x < 1,$

   $u(x,0) = 0, \quad u_t(x,0) = 1 - x, \quad u(0,t) = t, \quad u(1,t) = 0;$

4. $u_{tt} - 7u_t = u_{xx} + 2u_x - 2t - 7x - e^{-x}\sin 3x, \quad 0 < x < \pi,$

   $u(x,0) = 0, \quad u_t(x,0) = x, \quad u(0,t) = 0;$

5. $u_{tt} = u_{xx} + 4u + 2\sin^2 x, \quad 0 < x < \pi,$

   $u(x,0) = u_t(x,0) = 0, \quad u_x(0,t) = u_x(\pi,t) = 0;$

6. $u_{tt} - 3u_t = u_{xx} + u - x(4+t) + \cos\frac{3x}{2}, \quad 0 < x < \pi,$

   $u(x,0) = u_t(x,0) = x, \quad u_x(0,t) = t+1, \quad u(\pi,t) = \pi(t+1);$

7. $u_{tt} - 3u_t = u_{xx} + 2u_x - 3x + 2t, \quad 0 < x < \pi,$

   $u(x,0) = e^{-x}\sin x, \quad u_t(x,0) = x, \quad u(0,t) = 0, \quad u(\pi,t) = \pi t.$

Solution of Problem 2. Denote

$$w(x,t) = t(2-x).$$

The replacement $u(x,t) = v(x,t) + w(x,t)$ yields the following mixed problem with respect to $v(x,t)$:

$$v_{tt} = v_{xx} + v + w, \quad 0 < x < 2,\ t > 0,$$

$$v_{|x=0} = 0, \quad v_{|x=2} = 0,$$

$$v_{|t=0} = 0, \quad v_{t|t=0} = x - 2.$$

We seek the solution of this problem of the form

$$v(x,t) = \sum_{n=1}^{\infty} v_n(t) \sin\frac{n\pi}{2}x.$$

We have

$$x - 2 = \sum_{n=1}^{\infty} \gamma_n \sin\frac{n\pi}{2}x,$$

where

$$\gamma_n = \int_0^2 (x-2)\sin\frac{n\pi}{2}x\,dx = \frac{4}{n\pi}.$$

The functions $v_n(t)$ are solutions of the Cauchy problems

$$\ddot{v}_n(t) = \left(\left(\frac{n\pi}{2}\right)^2 - 1\right) v_n(t) = -t\gamma_n,$$

$$v_n(0) = 0, \quad \dot{v}_n(0) = \gamma_n.$$

Hence

$$v_n(t) = \gamma_n \frac{\sin \mu_n t}{\mu_n} - \int_0^t \frac{\sin \mu_n (t-\tau)}{\mu_n} \tau \gamma_n \, d\tau$$
$$= \frac{4}{n\pi}\left(\frac{\sin \mu_n t}{\mu_n} + \frac{t}{\mu_n^2} + \frac{\sin \mu_n t}{\mu_n^3}\right),$$
$$\mu_n = \sqrt{\left(\frac{n\pi}{2}\right)^2 - 1},$$

and consequently,

$$u(x,t) = t(2-x) + \sum_{n=1}^{\infty} \frac{4}{n\pi}\left(\frac{\sin \mu_n t}{\mu_n} + \frac{t}{\mu_n^2} + \frac{\sin \mu_n t}{\mu_n^3}\right) \sin \frac{n\pi}{2} x.$$

3. The Riemann method

6.2.23. Using the Riemann method (see Section 2.4) solve the following problems in the domain $-\infty < x < \infty$, $t > 0$:

1. $u_{xx} - u_{tt} = t, \quad u(x,0) = 0, \quad u_t(x,0) = x;$
2. $u_{xx} - u_{tt} = 2, \quad u(x,0) = \sin x, \quad u_t(x,0) = x^2;$
3. $u_{xx} - u_{tt} + 2u_x + u = e^{-x}, \quad u(x,0) = x, \quad u_t(x,0) = 1;$
4. $u_{xx} - u_{tt} + 2u_t - u = e^t, \quad u(x,0) = x, \quad u_t(x,0) = 0.$

Solution of Problem 1. The Riemann function satisfies the conditions

$$v_{\xi\xi} - v_{\eta\eta} = 0 \quad \text{in} \quad \Delta_{PQM},$$
$$v = 1 \quad \text{on} \quad MP,$$
$$v = 1 \quad \text{on} \quad MQ.$$

Consequently, $v \equiv 1$. Using the Riemann formula we get

$$u(x_0,t_0) = \frac{1}{2}\int_{x_0-t_0}^{x_0+t_0} x\,dx - \frac{1}{2}\int_0^{t_0}\left(\int_{t-t_0+x_0}^{-t+t_0+x_0} dx\right) t\,dt$$
$$= -\frac{1}{4}\left((x_0+t_0)^2 - (x_0-t_0)^2\right) - \frac{1}{2}\int_0^{t_0} 2t(t_0-t)\,dt$$
$$= x_0 t_0 + \left(t_0\frac{t^2}{2} - \frac{t^3}{3}\right)\Bigg|_0^{t_0} = x_0 t_0 - \left(\frac{t_0^3}{2} - \frac{t_0^3}{3}\right) = x_0 t_0 - \frac{t_0^3}{6}.$$

Hence,

$$u(x,t) = xt - \frac{t^3}{6}.$$

6.2.24. Find the Riemann function for the operator

$$\mathcal{L}u = u_{tt} - a^2 u_{xx}, \quad a = const,$$

and solve the Cauchy problem

$$u_{tt} = a^2 u_{xx} + f(x,t), \quad -\infty < x < \infty, \quad t > 0,$$

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x).$$

6.2.25. Solve the Cauchy problem

$$x^2 u_{xx} - y^2 u_{yy} = 0, \quad -\infty < x < \infty, \quad 1 < y < +\infty,$$

$$u|_{y=1} = \varphi(x), \quad u_y|_{y=1} = \psi(x).$$

6.3. Parabolic Partial Differential Equations

Parabolic partial differential equations usually describe various diffusion processes. The most important equation of parabolic type is the *heat (conduction) equation* or *diffusion equation*. In this section we suggest exercises for this type of equations.

1. The Cauchy problem for the heat equation

6.3.1. Using the Poisson formula (3.2.6) solve the following problems:

1. $u_t = \frac{1}{4} u_{xx}, \quad u(x,0) = e^{2x-x^2}, \quad -\infty < x < \infty,\ t \geq 0;$
2. $u_t = \frac{1}{4} u_{xx}, \quad u(x,0) = e^{-x^2} \sin x, \quad -\infty < x < \infty,\ t \geq 0;$
3. $u_t = u_{xx}, \quad u(x,0) = xe^{-x^2}, \quad -\infty < x < \infty,\ t \geq 0.$

6.3.2. Prove that the non-homogeneous equation

$$u_t = a^2 u_{xx} + f(x,t), \quad -\infty < x < \infty,\ t > 0$$

with the initial condition

$$u(x,0) = 0$$

has the following solution:

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} f(\xi,\tau) \frac{1}{2a\sqrt{\pi(t-\tau)}} \exp\Big(-\frac{(\xi-x)^2}{4a^2(t-\tau)}\Big)\, d\xi d\tau.$$

Hint. Apply the method from Chapter 2, Section 2.1 for the non-homogeneous wave equation.

6.3.3. Solve the following problems in the domain $-\infty < x < \infty,\ t \geq 0$:

1. $u_t = 4u_{xx} + t + e^t, \quad u(x,0) = 2;$
2. $u_t = u_{xx} + 3t^2, \quad u(x,0) = \sin x;$
3. $u_t = u_{xx} + e^{-t}\cos x, \quad u(x,0) = \cos x;$
4. $u_t = u_{xx} + e^t \sin x, \quad u(x,0) = \sin x;$
5. $u_t = u_{xx} + \sin t, \quad u(x,0) = e^{-x^2}.$

2. The mixed problem for the heat equation

Consider the following problem for the heat equation

$$\rho(x)u_t = (k(x)u_x)_x - q(x)u, \tag{6.3.1}$$

$$\rho(x) > 0, \quad k(x) > 0, \quad 0 < x < l,\ t > 0,$$

with the boundary conditions

$$(h_1 u_x + hu)|_{x=0} = (H_1 u_x + Hu)|_{x=l} = 0. \tag{6.3.2}$$

and with the initial condition

$$u_{|t=0} = \varphi(x). \tag{6.3.3}$$

For solving problem (6.3.1)-(6.3.3) we apply the method of separation of variables. We will seek non-trivial (i.e. not identically equal to zero) solutions of equation (6.3.1) satisfying the boundary conditions (6.3.2) and having the form $u(x,t) = T(t)y(x)$. Substituting this into (6.3.1) and (6.3.2) we obtain the equation

$$\dot{T}(t) + \lambda T(t) = 0$$

for the function $T(t)$, and the Sturm-Liouville problem L

$$-(k(x)y'(x))' + q(x)y(x) = \lambda\rho(x)y(x), \quad 0 \le x \le l,$$

$$h_1 y'(0) + hy(0) = H_1 y'(l) + Hy(l) = 0,$$

for the function $y(x)$. Let $\{\lambda_n\}$ and $\{y_n(x)\}$ be eigenvalues and eigenfunctions of the Sturm-Liouville problem respectively. Then the functions

$$u_n(x,t) = A_n \exp(-\lambda_n t) y_n(x)$$

satisfy (6.3.1) and (6.3.2) for any A_n .

We will seek the solution of the mixed problem (6.3.1)-(6.3.3) in the form

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\lambda_n t} y_n(x).$$

Using the initial condition (6.3.3) for finding the coefficients A_n, we get

$$\varphi(x) = \sum_{n=1}^{\infty} A_n y_n(x),$$

and consequently,

$$A_n = \int_0^l \rho(x)\varphi(x)y_n(x)\,dx.$$

6.3.4. Find the temperature distribution $u(x,t)$ in a slender homogeneous bar $(0 < x < l)$ with isolated lateral surface provided that the end-points $x = 0$ and $x = l$ are maintained at the temperature zero, and the initial temperature $u(x,0)$ is given:

1. $u(x,0) = u_0$;
2. $u(x,0) = u_0 x(l-x)$;
3. $u(x,0) = x \quad \left(0 \le x \le \frac{l}{2}\right)$,

   $u(x,0) = l - x \quad \left(\frac{l}{2} \le x \le l\right)$;
4. $u(x,0) = \sin\frac{\pi x}{l}$.

6.3.5. Find the temperature distribution $u(x,t)$ in a slender homogeneous bar $(0 < x < l)$ with isolated lateral surface and the isolated end-points $x = 0$ and $x = l$ provided that the initial temperature $u(x,0)$ is given:

1. $u(x,0) = u_0$;
2. $u(x,0) = u_0 \quad \left(0 \le x \le \frac{l}{2}\right), \quad u(x,0) = 0 \quad \left(\frac{l}{2} \le x \le l\right)$;
3. $u(x,0) = \sin\frac{\pi x}{l}$.
4. $u(x,0) = x \quad \left(0 \le x \le \frac{l}{2}\right), \quad u(x,0) = l - x \quad \left(\frac{l}{2} \le x \le l\right)$.

Solution of Problem 3. We have to solve the following mixed problem

$$u_t = a^2 u_{xx}, \quad 0 < x < l,\ t > 0,$$

$$u_{x|x=0} = u_{x|x=l} = 0,$$

$$u_{|t=0} = \sin\frac{\pi}{l}x.$$

First we seek nontrivial particular solutions of the equation of the form $u(x,t) = Y(x)T(t)$, which satisfy the boundary conditions. Repeating the arguments from section 3.1 we obtain for the function $T(t)$ the ordinary differential equation

$$\dot{T}(t) + a^2\lambda T(t) = 0,$$

and for the function $Y(x)$ we get the Sturm-Liouville boundary value problem

$$Y''(x) + \lambda Y(x) = 0, \quad Y'(0) = Y'(l) = 0.$$

Here λ is the spectral parameter. The eigenvalues and the eigenfunctions of this boundary value problem are

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad Y_n(x) = \cos\frac{n\pi}{l}x, \quad n \geq 0.$$

The corresponding equations

$$\dot{T}_n(t) + a^2\lambda_n T_n(t) = 0, \quad ,n \geq 0,$$

have the general solutions

$$T_n(t) = A_n \exp\left(-\left(\frac{an\pi}{l}\right)^2 t\right), \ n \geq 0,$$

where A_n are arbitrary constants. Hence,

$$u_n(x,t) = A_n \exp\left(-\left(\frac{an\pi}{l}\right)^2 t\right) \cos\frac{n\pi}{l}x, \quad n \geq 0.$$

We seek the solution of the mixed problem of the form

$$u(x,t) = \sum_{n=0}^{\infty} A_n \exp\left(-\left(\frac{an\pi}{l}\right)^2 t\right) \cos\frac{n\pi}{l}x,$$

and we find A_n from the initial conditions:

$$\sin\frac{\pi}{l}x = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi}{l}x,$$

hence

$$A_0 = \frac{1}{l}\int_0^l \sin\frac{\pi}{l}x\,dx,$$

$$A_n = \frac{2}{l}\int_0^l \sin\frac{\pi}{l}x\cos\frac{n\pi}{l}x\,dx \quad n \geq 1,$$

We calculate

$$A_0 = \frac{2}{\pi}, \quad A_{2n+1} = 0, \quad A_{2n} = \frac{4}{\pi(1-4n^2)},$$

and consequently,

$$u(x,t) = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4}{\pi(1-4n^2)} \exp\left(-\left(\frac{2an\pi}{l}\right)^2 t\right) \cos\frac{2n\pi}{l}x.$$

6.3.6. Solve the following problems applying the method of separation of variables:

1. $u_t = u_{xx}, \quad 0 < x < 1,$
   $u_x(0,t) = u(1,t) = 0, \quad u(x,0) = x^2 - 1;$
2. $u_{xx} = u_t + u, \quad 0 < x < l,$
   $u(0,t) = u(l,t) = 0, \quad u(x,0) = u_0;$
3. $u_t = u_{xx} - 4u, \quad 0 < x < \pi,$
   $u(0,t) = u(\pi,t) = 0, \quad u(x,0) = x^2 - \pi x;$
4. $u_t = a^2 u_{xx} - \beta u, \quad 0 < x < l,$
   $u(0,t) = u_x(l,t) = 0, \quad u(x,0) = \sin\dfrac{\pi x}{2l};$

Solution of Problem 2. We seek the solution in the form

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\lambda_n t} y_n(x),$$

where the numbers λ_n are the eigenvalues, and the functions $y_n(x)$ are the eigenfunctions of the boundary value problem

$$y'' + (\lambda - 1)y = 0, \quad y(0) = y(l) = 0.$$

Solving this boundary value problem we get

$$\lambda_n = \left(\frac{\pi n}{l}\right)^2 + 1,$$

$$y_n(x) = \sin\frac{\pi n}{l}x, \quad n \geq 1.$$

Therefore,

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\left(1+(\frac{\pi n}{l})^2\right)t} \sin\frac{\pi n}{l}x.$$

Using the initial conditions we obtain

$$\sum_{n=1}^{\infty} A_n \sin\frac{\pi n}{l}x = u_0,$$

and consequently

$$A_n = \frac{u_0 l}{\pi n}(1 + (-1)^{n+1}).$$

The solution of Problem 2 has the form

$$u(x,t) = \sum_{n=1}^{\infty} \frac{u_0 l}{\pi n}(1 + (-1)^{n+1}) e^{-\left(1+(\frac{\pi n}{l})^2\right)t} \sin\frac{\pi n}{l}x.$$

Using the method of separation of variables one can solve also problems for non-homogeneous equations and boundary conditions. Consider the following mixed problem

$$\rho(x)u_t = (k(x)u_x)_x - q(x)u + f(x,t), \quad 0 < x < l,\ t > 0, \tag{6.3.4}$$

$$(h_1 u_x + hu)|_{x=0} = \mu(t), \quad (H_1 u_x + Hu)|_{x=l} = \nu(t), \tag{6.3.5}$$

$$u|_{t=0} = \varphi(x). \tag{6.3.6}$$

In the case of homogeneous boundary conditions $(\mu(t) = \nu(t) = 0)$ we seek the solution of (6.3.4)-(6.3.6) in the form of a series

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) y_n(x)$$

with respect to the eigenfunctions of the corresponding problem L. Substituting this into (6.3.4) and (6.3.5) we obtain a Cauchy problem for $T_n(t)$:

$$\dot{T}_n(t) + \lambda_n T_n(t) = f_n(t), \quad T_n(0) = \varphi_n, \tag{6.3.7}$$

where $f_n(t)$ and φ_n are the Fourier coefficients for the functions

$$\frac{1}{\rho(x)} f(x,t) \quad \text{and} \quad \varphi(x),$$

respectively:

$$f_n(t) = \int_0^l f(x,t) y_n(x)\,dx, \quad \varphi_n = \int_0^l \rho(x)\varphi(x) y_n(x)\,dx.$$

Solving the Cauchy problem (6.3.7) we calculate

$$T_n(t) = \varphi_n e^{-\lambda_n t} + \int_0^t f_n(\tau) e^{-\lambda_n (t-\tau)}\,d\tau.$$

Problem (6.3.4)-(6.3.6) with non-homogeneous boundary conditions can be reduced to the problem with homogeneous boundary conditions with the help of the replacement of the unknown function $u(x,t) = v(x,t) + w(x,t)$, where the function $w(x,t)$ is chosen such that it satisfies the boundary conditions (6.3.5).

6.3.7. Find the temperature distribution $u(x,t)$ in a slender homogeneous bar $(0 < x < l)$ with isolated lateral surface provided that the end-points $x = 0$ and $x = l$ are maintained at the constant temperatures $u(0,t) = \mu_0$, $u(l,t) = \nu_0$, and where the initial temperature is $u(x,0) = \varphi(x)$.

6.3.8. The heat radiation takes place from the lateral surface of a slender homogeneous bar $(0 < x < l)$ to the surrounding medium of zero temperature. Find the temperature distribution $u(x,t)$ if the end-point s $x = 0$ and $x = l$ are maintained at the constant temperatures $u(0,t) = \mu_0$, $u(l,t) = \nu_0$, and if the initial temperature is $u(x,0) = \varphi(x)$.

6.3.9. Solve the following problems applying the method of separation of variables:

1. $u_t = a^2 u_{xx}, \quad 0 < x < l,$
   $u(0,t) = 0, \quad u(l,t) = At, \quad u(x,0) = 0;$
2. $u_t = u_{xx}, \quad 0 < x < l,$
   $u_x(0,t) = 1, \quad u(l,t) = 0, \quad u(x,0) = 0;$
3. $u_t = u_{xx}, \quad 0 < x < l,$
   $u_x(0,t) = u_x(l,t) = u_0, \quad u(x,0) = Ax;$
4. $u_t = u_{xx}, \quad 0 < x < l,$
   $u_x(0,t) = At, \quad u_x(l,t) = B, \quad u(x,0) = 0;$
5. $u_t = a^2 u_{xx} - \beta u, \quad 0 < x < l,$
   $u(0,t) = \mu_0, \quad u_x(l,t) = \nu_0, \quad u(x,0) = 0;$
6. $u_t = a^2 u_{xx} - \beta u + \sin \dfrac{\pi x}{l}, \quad 0 < x < l,$
   $u(0,t) = u(l,t) = 0, \quad u(x,0) = 0;$
7. $u_t = u_{xx} + u - x + 2\sin 2x \cos x, \quad 0 < x < \dfrac{\pi}{2},$
   $u(0,t) = 0, \quad u_x(\dfrac{\pi}{2},t) = 1, \quad u(x,0) = x;$
8. $u_t = u_{xx} + u + xt(2-t) + 2\cos t, \quad 0 < x < \pi,$
   $u_x(0,t) = u_x(\pi,t) = t^2, \quad u(x,0) = \cos 2x;$
9. $u_t = u_{xx} - 2u_x + u + e^x \sin x - t, \quad 0 < x < \pi,$
   $u(0,t) = u(\pi,t) = 1 + t, \quad u(x,0) = 1 + e^x \sin 2x;$
10. $u_t = u_{xx} - 2u_x + x + 2t, \quad 0 < x < 1,$
    $u(0,t) = u(1,t) = t, \quad u(x,0) = e^x \sin \pi x.$

Solution of Problem 1. By the replacement

$$u(x,t) = v(x,t) + \frac{x}{l}At$$

we reduce our problem to the following mixed problem with respect to $v(x,t)$:

$$v_t = a^2 v_{xx} - \frac{Ax}{l}, \quad 0 < x < l,\ t > 0,$$

$$v(0,t) = v(l,t) = 0, \quad v(x,0) = 0.$$

We seek the solution of this problem in the form

$$v(x,t) = \sum_{n=1}^{\infty} v_n(t) \sin \frac{\pi n}{l} x.$$

We have

$$\frac{Ax}{l} = \sum_{n=1}^{\infty} f_n \sin \frac{\pi n}{l} x,$$

where

$$f_n = \frac{2A}{l^2} \int_0^l x \sin \frac{\pi n}{l} x \, dx = \frac{2A}{n\pi} (-1)^{n-1}.$$

The functions $v_n(t)$ are solutions of the Cauchy problems

$$\dot{v}_n(t) + \left(\frac{an\pi}{l} \right)^2 v_n(t) = (-1)^n \frac{2A}{n\pi},$$

$$v_n(0) = 0.$$

Hence

$$v_n(t) = (-1)^n \frac{2Al^2}{a^2 n^3 \pi^3} \left(1 - \exp\left(-\left(\frac{an\pi}{l} \right)^2 \right) t \right),$$

and consequently,

$$u(x,t) = \frac{x}{l} At + \sum_{n=1}^{\infty} (-1)^n \frac{2Al^2}{a^2 n^3 \pi^3} \left(1 - \exp\left(-\left(\frac{an\pi}{l} \right)^2 \right) t \right) \sin \frac{\pi n}{l} x.$$

Solution of Problem 5. Denote

$$w(x) = \left(1 - \frac{x}{l} \right)^2 \mu_0 + \frac{x^2}{2l} \nu_0.$$

Clearly, $w(0) = \mu_0$, $w'(l) = \nu_0$. The replacement $u(x,t) = v(x,t) + w(x)$ yields the following mixed problem with respect to $v(x,t)$:

$$v_t = a^2 v_{xx} - \beta v + f(x), \quad 0 < x < l, \; t > 0,$$

$$v(0,t) = v_x(l,t) = 0, \quad v(x,0) = -w(x),$$

where $f(x) = a^2 w''(x) - \beta w(x)$. We seek the solution of this problem of the form

$$v(x,t) = \sum_{n=0}^{\infty} v_n(t) \sin \left(n + \frac{1}{2} \right) \frac{\pi}{l} x.$$

The functions $v_n(t)$ are solutions of the Cauchy problems

$$\dot{v}_n(t) + \left(\frac{a(2n+1)\pi}{2l} \right)^2 v_n(t) - \beta v_n(t) = f_n,$$

$$v_n(0) = -w_n,$$

where

$$f_n = \frac{2}{l}\int_0^l f(x)\sin\left(n+\frac{1}{2}\right)\frac{\pi}{l}x\,dx, \quad w_n = \frac{2}{l}\int_0^l w(x)\sin\left(n+\frac{1}{2}\right)\frac{\pi}{l}x\,dx.$$

Hence

$$v_n(t) = \frac{f_n}{\lambda_n} - \left(w_n + \frac{f_n}{\lambda_n}\right)\exp(-\lambda_n t),$$

and consequently,

$$u(x,t) = w(x) + \sum_{n=0}^{\infty}\left(\frac{f_n}{\lambda_n} - \left(w_n + \frac{f_n}{\lambda_n}\right)\exp(-\lambda_n t)\right)\sin\left(n+\frac{1}{2}\right)\frac{\pi}{l}x,$$

where

$$\lambda_n = \left(\frac{a(2n+1)\pi}{2l}\right)^2 - \beta.$$

Solution of Problem 10. By the replacement $u(x,t) = v(x,t) + t$ we reduce our problem to the following mixed problem with respect to $v(x,t)$:

$$v_t = v_{xx} - 2v_x + x + 2t - 1, \quad 0 < x < 1,\ t > 0,$$

$$v(0,t) = v(1,t) = 0, \quad v(x,0) = e^x \sin \pi x.$$

We seek the solution of this problem of the form

$$v(x,t) = e^x \sum_{n=1}^{\infty} v_n(t)\sin \pi n x.$$

The functions $v_n(t)$ are solutions of the Cauchy problems

$$\dot{v}_n(t) + \left((n\pi)^2 + 1\right)v_n(t) = f_n(t),$$

$$v_n(0) = \gamma_n,$$

where

$$f_n(t) = 2\int_0^1 e^{-x}(x + 2t - 1)\sin \pi n x\,dx, \quad n \geq 1,$$

$$\gamma_1 = 1, \quad \gamma_n = 0,\ n \geq 2.$$

Hence

$$v_n(t) = \gamma_n \exp(-(n^2\pi^2 + 1)t) + \int_0^t f_n(\tau)\exp(-(n^2\pi^2 + 1)(t - \tau))\,d\tau.$$

6.4. Elliptic Partial Differential Equations

Elliptic equations usually describe stationary fields, for example, gravitational, electrostatical and temperature fields. The most important equations of elliptic type are the Laplace equation $\Delta u = 0$ and the Poisson equation $\Delta u = f(x)$, where $x = (x_1, \ldots, x_n)$ are spatial variables, $u(x)$ is an unknown function,

$$\Delta u := \sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2}$$

is the Laplace operator (or Laplacian), and $f(x)$ is a given function. In this section we suggest exercises for boundary value problems for elliptic partial differential equations.

We will need the expressions of the Laplace operator in curvilinear coordinates:
i) polar coordinates $(n = 2, x_1 = \rho\cos\varphi, x_2 = \rho\sin\varphi)$:

$$\Delta u(\rho,\varphi) = \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \rho^2};$$

ii) cylindrical coordinates $(n = 3,\ x_1 = \rho\cos\varphi,\ x_2 = \rho\sin\varphi,\ x_3 = z)$:

$$\Delta u(\rho,\varphi,z) = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2};$$

iii) spherical coordinates $(n = 3, x_1 = r\cos\varphi\sin\theta, x_2 = r\sin\varphi\cos\theta, z = r\cos\theta)$:

$$\Delta u(r,\varphi,\theta) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2}.$$

1. Harmonic functions and their properties

A function $u(x) = u(x_1, \ldots, x_n)$ is called *harmonic* in a domain $D \subset \mathbf{R^n}$, if $u(x) \in C^2(D)$ (i.e. twice continuously differentiable in D) and $\Delta u = 0$ in D. If D is unbounded then there is the additional condition: $u(x)$ is bounded for $n = 2$, and $u(x) \to 0$ for $n \geq 3$ as $|x| := \sqrt{\sum_{i=1}^{n} x_i^2} \to \infty$.

6.4.1. Check that the following functions are harmonic:

1. $u_1(x_1,x_2) = \ln\frac{1}{|x-y|} = \ln\frac{1}{\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}}$,
   $n = 2,\ x = (x_1,x_2),\ y = (y_1,y_2),\ x \neq y;$
2. $u_2(x_1,x_2,x_3) = \frac{1}{|x-y|} = \frac{1}{\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2}}$,
   $n = 3,\ x = (x_1,x_2,x_3),\ y = (y_1,y_2,y_3),\ x \neq y,$

where y is a fixed point.

The function u_1 is called the *fundamental solution* of the Laplace equation on the plane. The function u_2 is called the *fundamental solution* of the Laplace equation in the space $\mathbf{R}^3$. The importance of the fundamental solution connects with the isotropy of the space when the physical picture depends only on the distance from the source of energy but not on the direction. For example, the function u_2 represents the potential of the gravitational (electro-statical) field created by a point unit mass (point unit charge). Similar sense has the fundamental solution of the Laplace equation on the plane: this is the potential of the field produced by a charge of constant linear density $q=1,$ distributed uniformly along the line $x_1=y_1, x_2=y_2$.

6.4.2. Find all harmonic functions, defined in a bounded domain $D\subset\mathbf{R^2},$ and having the following form:

1. $u(x_1,x_2)$ is a polynomial of the second degree;
2. $u(x_1)$ depends only on one variable;
3. $u(x_1,x_2)$ depends only on the distance $\rho=\sqrt{x_1^2+x_2^2}$;
4. $u(x_1,x_2)$ depends only on the angle φ between the x_1-axis and the vector connecting the origin and the point $x=(x_1,x_2)$.

6.4.3. Find all harmonic functions, defined in a bounded domain $D\subset\mathbf{R^3},$ and having the following form:

1. $u(x_1,x_2,x_3)$ is a polynomial of the second degree;
2. $u(x_1,x_2,x_3)$ depends only on the distance $r=\sqrt{x_1^2+x_2^2+x_3^2}$;
3. $u(x_1,x_2,x_3)$ depends only on the angle θ between the coordinate x_1x_2-plane (i.e. $x_3=0$) and the vector connecting the origin and the point $x=(x_1,x_2,x_3)$.

6.4.4. Find the value of the constant $k,$ for which the following functions are harmonic in a bounded domain $D\subset\mathbf{R^2}$:

1. $u(x_1,x_2)=x_1^3+kx_1x_2^2$;
2. $u(x_1,x_2)=e^{2x_1}\cosh kx_2$;
3. $u(x_1,x_2)=\sin 3x_1\cosh kx_2$.

6.4.5. Let the function $u(x_1,x_2)$ be harmonic in a bounded domain $D\subset\mathbf{R^2}$. Check which of these functions are harmonic:

1. $\dfrac{\partial u(x_1,x_2)}{\partial x_1}\cdot\dfrac{\partial u(x_1,x_2)}{\partial x_2}$;
2. $x_1\dfrac{\partial u(x_1,x_2)}{\partial x_1}-x_2\dfrac{\partial u(x_1,x_2)}{\partial x_2}$;

3. $x_2 \dfrac{\partial u(x_1,x_2)}{\partial x_1} - x_1 \dfrac{\partial u(x_1,x_2)}{\partial x_2}$.

6.4.6. Let the function $u(x_1,x_2,x_3)$ be harmonic in a bounded domain $D \subset \mathbf{R^3}$. Check which of these functions are harmonic:

1. $u(ax_1,ax_2,ax_3)$, where a is a constant;
2. $u(x_1+h_1,x_2+h_2,x_3+h_3)$, where h_1,h_2,h_3 are constants;
3. $\dfrac{\partial u(x_1,x_2,x_3)}{\partial x_1} \cdot \dfrac{\partial u(x_1,x_2,x_3)}{\partial x_2}$.

6.4.7. Can a non-constant harmonic function in a bounded domain D have a closed level curve?

Hint. Use the maximum principle for harmonic functions: Let $u(x)$, $x \in \mathbf{R^n}$ be harmonic in a domain $D \subset \mathbf{R^n}$ and continuous in $\overline{D}$. Then $u(x)$ attains its maximum and minimum on the boundary of D.

The theory of harmonic functions of two variables has a deep connection with the theory of analytic functions. Let $n=2$, i.e. $x=(x_1,x_2)$, and let $z=x_1+ix_2$. If the function $f(z)=u(x_1,x_2)+iv(x_1,x_2)$ is analytic in D, then by virtue of the Cauchy-Riemann differential equations

$$\frac{\partial u}{\partial x_1}=\frac{\partial v}{\partial x_2}, \quad \frac{\partial u}{\partial x_2}=-\frac{\partial v}{\partial x_1},$$

the functions u and v are harmonic in D (and they are called conjugate harmonic functions). Conversely, if $u(x_1,x_2)$ is harmonic in the simply connected domain D, then there exists a conjugate harmonic function $v(x_1,x_2)$ such that the function $f=u+iv$ is analytic in D, and

$$v(x_1,x_2)=\int_{(y_1,y_2)}^{(x_1,x_2)} \left(-\frac{\partial u}{\partial x_2}\,dx_1+\frac{\partial u}{\partial x_1}\,dx_2\right)+C, \tag{6.4.1}$$

where (y_1,y_2) is a fixed point in D (the integral does not depend on the way of the integration since under the integral we have the total differential of v).

Another way of recovering of an analytic function from its real part is given by the Goursat formula:

$$f(z)=2u\left(\frac{z}{2},\frac{z}{2i}\right)-u(0,0)+iC, \quad z=x_1+ix_2, \quad (0,0)\in D. \tag{6.4.2}$$

6.4.8. Let the function $u(x_1,x_2)$ be harmonic. Show that the function

$$f(z)=\frac{\partial u(x_1,x_2)}{\partial x_1}-i\frac{\partial u(x_1,x_2)}{\partial x_2}, \quad z=x_1+ix_2$$

is analytic.

6.4.9. Let the function $u(x_1,x_2)$ be harmonic in D with $(u_{x_1})^2+(u_{x_2})^2 \neq 0$ in D. Show that the function

$$\frac{u_{x_1}}{(u_{x_1})^2+(u_{x_2})^2}$$

is also harmonic in D.

6.4.10. Recover function $f(z)$, which is analytic in a simply connected domain D, from its real part:

1. $u(x_1,x_2)=x_1^3-3x_1x_2^2$;
2. $u(x_1,x_2)=e^{x_1}\sin x_2$;
3. $u(x_1,x_2)=\sin x_1\cosh x_2$;
4. $u(x_1,x_2)=x_1^2-x_2^2$;
5. $u(x_1,x_2)=x_1x_2+x_1+x_2$.

Solution of Problem 2. We construct the conjugate function $v(x_1,x_2)$ by (6.4.1). As path of integration we choose the rectilinear segments connecting the points $(y_1,y_2),(x_1,y_2)$ and $(x_1,y_2),(x_1,x_2)$ (we suppose that these segments lie in D). Then

$$v(x_1,x_2)=-\int_{y_1}^{x_1}e^t\cos y_2\,dt+\int_{y_2}^{x_2}e^{x_1}\sin t\,dt+C$$
$$=e^{y_1}\cos y_2-e^{x_1}\cos x_2+C.$$

Hence

$$f(z)=e^{x_1}\sin x_2-ie^{x_1}\cos x_2+i(e^{y_1}\cos y_2+C).$$

6.4.11. Show that if two harmonic functions in D coincide in a domain $D_0\subset D$, then they coincide in the whole domain D.

6.4.12. Let $u(x_1,x_2)$ be a harmonic function. Calculate the integrals:

1. $\int_S u(x_1^2-x_2^2,2x_1,x_2)\,ds,\quad S=\{(x_1,x_2):x_1^2+x_2^2=1\};$
2. $\int_0^{2\pi} u\Big((\sin\cos\varphi)(\cosh\sin\varphi),(\cos\cos\varphi)(\sinh\sin\varphi)\Big)\,d\varphi;$
3. $\int_0^{2\pi} u\left(\sum_{k=1}^N\cos k\varphi,\sum_{k=1}^N\sin k\varphi\right)\,d\varphi.$

Solution of Problem 3. We use the following important property of harmonic functions: If the function $u(x_1,x_2)$ is harmonic in D and

$$z=g(\zeta)=\nu(\xi,\eta)+i\mu(\xi,\eta)$$

is an analytic function in a domain Δ having values in D, then the composite function

$$u(\nu(\xi,\eta),\mu(\xi,\eta))=U(\xi,\eta)$$

is harmonic in D with respect to (ξ,η).

We introduce the function

$$U(\xi,\eta)=u\Big(\Re\sum_{k=1}^{N}\zeta^k,\Im\sum_{k=1}^{N}\zeta^k\Big),\quad \zeta=\xi+i\eta.$$

Clearly, U is harmonic. Consider this function on the circle $\xi^2+\eta^2=1$. Let us introduce the polar coordinates $\xi=\cos\varphi,\eta=\sin\varphi$. Then $\zeta^k=\cos k\varphi+i\sin k\varphi$,

$$\Re\sum_{k=1}^{N}\zeta^k=\sum_{k=1}^{N}\cos k\varphi,\quad \Im\sum_{k=1}^{N}\zeta^k=\sum_{k=1}^{N}\sin k\varphi.$$

Therefore,

$$\int_0^{2\pi}U(\cos\varphi,\sin\varphi)\,d\varphi=\int_0^{2\pi}u\left(\sum_{k=1}^{N}\cos k\varphi,\sum_{k=1}^{N}\sin k\varphi\right)d\varphi.$$

By the mean-value theorem for harmonic functions,

$$\int_0^{2\pi}U(\cos\varphi,\sin\varphi)\,d\varphi=2\pi U(0,0).$$

Since $\sum_{k=1}^{N}\zeta^k|_{\zeta=0}=0$, we have $U(0,0)=u(0,0)$. Hence the desired integral is equal to $2\pi u(0,0)$.

2. Boundary value problems for elliptic equations. The Green's function

Let $D\subset\mathbf{R^n}$ be a bounded domain with a piecewise smooth boundary S.

Dirichlet problem. Let a continuous function $g(x)$ be given on S. Find a function $u(x)$ which is harmonic in D and continuous in $\overline{D}$ and has on S assigned values:

$$\Delta u=0\quad (x\in D),$$

$$u_{|S}=g.$$

Neumann problem. Let a continuous function $g(x)$ be given on S. Find a function $u(x)$ which is harmonic in D and $u(x)\in C^1(\overline{D})$, $\frac{\partial u}{\partial n}_{|S}=g$, where n is the outer normal to S:

$$\Delta u=0\quad (x\in D),$$

$$\left.\frac{\partial u}{\partial n}\right|_S=g.$$

Then the Dirichlet and Neumann problems can be considered also in unbounded domains. In this case we need the additional condition $u(\infty)=0$ (for $n=2$ the additional condition has the form $u(x)=O(1)$, $|x|\to\infty$). For example, let $D\subset\mathbf{R}^3$ be a bounded set, and $D_1:=\mathbf{R}^3\setminus\overline{D}$. Then the Dirichlet and Neumann problems in the domain D_1 are called *exterior problems*.

Exterior Dirichlet problem:

$$\Delta u = 0 \quad (x \in D_1),$$

$$u_{|S} = g,\ u(\infty) = 0 \quad (u(x) = O(1),\, |x| \to \infty \ \text{for}\ n = 2).$$

Exterior Neumann problem:

$$\Delta u = 0 \quad (x \in D_1),$$

$$\frac{\partial u}{\partial n}\Big|_S = g,\ u(\infty) = 0, \quad (u(x) = O(1),\, |x| \to \infty \ \text{for}\ n = 2).$$

One can consider the Dirichlet and Neumann problems also in other unbounded regions (a sector, a strip, a half-strip, a half-plane, etc.). Analogously one can formulate the Dirichlet and Neumann problems for the Poisson equation

$$\Delta u(x) = -f(x).$$

The Dirichlet problem has a unique solution. If $n \geq 3$ then the exterior Neumann problem also has a unique solution. The solution of the interior Neumann problem (for $n = 2$ also of the exterior Neumann problem) is defined up to an additive constant. Note that

$$\int_S g(x)\, ds = 0$$

is a necessary condition for the solvability of the Neumann problem.

6.4.13. Find the solution of the Dirichlet problem

$$\Delta u(x_1, x_2) = 0, \quad x \in D,$$

$$u(x_1, x_2)|_S = g(x_1, x_2),$$

where $D = \{x : |x| < l\} \subset \mathbf{R^2}$ is the disc of radius l around the origin, and

1. $g(x_1, x_2) = a$;
2. $g(x_1, x_2) = ax_1 + bx_2$;
3. $g(x_1, x_2) = x_1 x_2$;
4. $g(\rho, \varphi) = a + b\sin\varphi$;
5. $g(\rho, \varphi) = a\sin^2\varphi + b\cos^2\varphi$.

Solution of Problem 4. Let us go on from polar coordinates to cartesian ones. If $(x_1, x_2) \in S$ then $x_1 = l\cos\varphi$, $x_2 = l\sin\varphi$, hence

$$g(x_1, x_2) = a + \frac{b}{l} x_2.$$

Since the function $A+Bx_2$ is harmonic it follows that the solution has the form

$$u(x_1,x_2)=a+\frac{b}{l}x_2$$

or

$$u(\rho,\varphi)=a+\frac{b}{l}\rho\sin\varphi.$$

Solution of Problem 2. Let us go on to polar coordinates. Then

$$g(\rho,\varphi)=al\cos\varphi+bl\sin\varphi.$$

We seek a solution of the form

$$u(\rho,\varphi)=A(\rho)\cos k_1\varphi+B(\rho)\sin k_2\varphi.$$

Then the functions $A(\rho)$ and $B(\rho)$ satisfy the Euler equations

$$\rho^2A''(\rho)+\rho A'(\rho)-k_1^2A(\rho)=0,$$

$$\rho^2B''(\rho)+\rho B'(\rho)-k_2^2B(\rho)=0.$$

The general solutions of these equations have the form

$$A(\rho)=\alpha_1\rho^{-|k_1|}+\beta_1\rho^{|k_1|},$$

$$B(\rho)=\alpha_2\rho^{-|k_2|}+\beta_2\rho^{|k_2|}.$$

In our case we have $k_1=k_2=1$. Since

$$|u(x_1,x_2)|<\infty,\quad u(l,\varphi)=g(l,\varphi),\quad 0\le\varphi\le 2\pi,$$

we get

$$\beta_1=\beta_2=0,\quad \alpha_1 l^{-1}=al,\quad \alpha_2 l^{-1}=bl.$$

Consequently, the solution has the form

$$u(\rho,\varphi)=\frac{al^2}{\rho}\cos\varphi+\frac{bl^2}{\rho}\sin\varphi.$$

6.4.14. Find the solution of the Dirichlet problem

$$\Delta u(x_1,x_2)=0,\quad x\in D,$$

$$u(x_1,x_2)|_S=g(x_1,x_2),$$

where $D=\{x:\,|x|>l\}\subset\mathbf{R^2}$ is the exterior of the disc of radius l around the origin, and the function $g(x_1,x_2)$ is taken from Problem 6.4.13 (1.-5.)

6.4.15. Find the solution of the Dirichlet problem

$$\Delta u(x_1,x_2)=0,\quad x\in D,$$

$$u(x_1,x_2)|_{|x|=l_1} = a, \quad u(x_1,x_2)|_{|x|=l_2} = b,$$

where $D = \{x:\ l_1 < |x| < l_2\} \subset \mathbf{R^2}$ is a ring.

6.4.16. Find the solution of the Dirichlet problem for the Poisson equation

$$\Delta u(x_1,x_2) = 1,$$

$$u(x_1,x_2)|_S = 0,$$

where $D = \{x:\ |x| < l\} \subset \mathbf{R^2}$ is the disc of radius l around the origin.

6.4.17. Find the solution of the Dirichlet problem

$$\Delta u(x_1,x_2,x_3) = 0, \quad x \in D,$$

$$u(x_1,x_2,x_3)|_S = a,$$

where $D = \{x:\ |x| < l\} \subset \mathbf{R^3}$ is the ball of radius l around the origin.

6.4.18. Find the solution of the Dirichlet problem

$$\Delta u(x_1,x_2,x_3) = 0, \quad x \in D,$$

$$u(x_1,x_2,x_3)|_S = a,$$

where $D = \{x:\ |x| > l\} \subset \mathbf{R^3}$ is the exterior of the ball of radius l.

6.4.19. Find the solution of the Neumann problem

$$\Delta u(x_1,x_2) = 0, \quad x \in D,$$

$$\left.\frac{\partial u(x_1,x_2)}{\partial n}\right|_S = g(x_1,x_2),$$

where $D = \{x : |x| < l\} \subset \mathbf{R^2}$ is the disc of radius l around the origin, and

1. $g(x_1,x_2) = a$;
2. $g(x_1,x_2) = ax_1$;
3. $g(x_1,x_2) = a(x_1^2 - x_2^2)$;
4. $g(\rho,\varphi) = a\cos\varphi + b$;
5. $g(\rho,\varphi) = a\sin\varphi + b\sin^3\varphi$.

Solution of Problem 2. First we check the condition

$$\int_S g(x)ds = 0.$$

Since

$$\int_S ax_1 ds = al\int_0^{2\pi} \cos\varphi\, d\varphi = 0,$$

it follows that the problem is solvable. We seek the solution in the form

$$u(x_1,x_2) = \alpha x_1,$$

and we define α from the boundary condition. Let us introduce polar coordinates, then

$$u(\rho,\varphi) = \alpha\rho\cos\varphi.$$

The differentiation with respect to the normal coincides with the differentiation with respect to ρ, hence

$$\frac{\partial u}{\partial n}|_S = al\cos\varphi = g = \alpha\cos\varphi.$$

Therefore $\alpha = al$, and consequently,

$$u(\rho,\varphi) = al\rho\cos\varphi + C$$

or

$$u(x_1,x_2) = alx_1 + C.$$

Let $D \subset \mathbf{R^3}$ be a bounded domain with the piecewise smooth boundary S. Fix $y \in D$. The function $G(x,y)$, $x \in \overline{D}$ is called the Green's function for D if

$$G(x,y) = \frac{1}{4\pi|x-y|} + v(x,y),$$

where the function $v(x,y)$ is harmonic for $x \in D$ and continuous for $x \in \overline{D}$, and such that

$$G(x,y)|_{x\in S} = 0.$$

If D is unbounded then there is the additional condition $v(x,y) \to 0$ as $|x| \to \infty$. Using the Green's function one can write the solution of the Dirichlet problem

$$\Delta u(x) = -f(x), \quad u(x)|_S = g(x)$$

in the form

$$u(x) = -\int_S \frac{\partial G(x,y)}{\partial n_y} g(y)\,dy + \int_D G(x,y)f(y)\,dy. \tag{6.4.3}$$

For constructing the Green's function for some symmetrical domains one can use the so-called reflection method. In this method we seek the function $v(x,y)$ as a sum of terms of the form $\dfrac{\alpha_i}{|x-y^i|}$ where the points y^i are chosen outside D and such that the Green's function satisfies the boundary condition.

For $n = 2$ the Green's function is defined analogously, namely:

(i) $G(x,y) = \dfrac{1}{2\pi}\ln\dfrac{1}{|x-y|} + v(x,y);$

(ii) $G(x,y)|_{x\in S} = 0; \quad x = (x_1,x_2) \in \overline{D}, \quad y = (y_1,y_2) \in D.$

Note that the Green's function $G(x,y)$ is symmetrical with respect to x and y:

$$G(x,y) = G(y,x).$$

6.4.20. Construct the Green's function for the following domains:
1) the ball of radius R around the origin, $n=3$;
2) the disc of radius R around the origin, $n=2$.

Solution of Problem 1. Choose the point $y'=(y'_1,y'_2,y'_3)$, symmetrically to the point $y=(y_1,y_2,y_3)$ with respect to the sphere S (i.e. $|y|\cdot|y'|=R^2$). Then

$$y'=\frac{R^2}{|y|^2}y.$$

We seek the function $v(x,y)$ in the form

$$v(x,y)=-\frac{\alpha}{4\pi|x-y'|}.$$

The Green's function is defined by the formula

$$G(x,y)=\frac{1}{4\pi|x-y|}-\frac{\alpha}{4\pi|x-y'|},$$

where α is a constant.

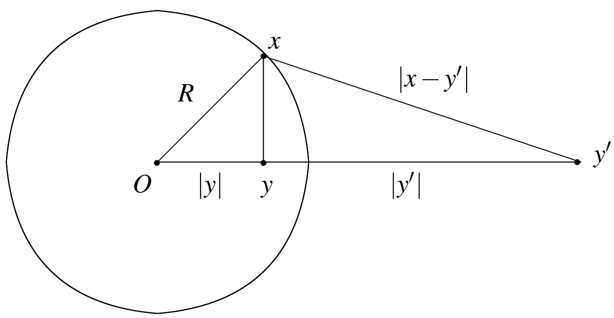

Figure 6.4.1.

Let $x\in S$ (see fig. 6.4.1). Since the triangles Oxy' and Oxy are similar it follows that

$$\frac{R}{|y|}=\frac{|x-y'|}{|x-y|},$$

and consequently,

$$\frac{1}{4\pi|x-y|}=\frac{1}{4\pi}\frac{R}{|y|}\frac{1}{|x-y'|}.$$

Hence, if

$$\alpha=\frac{R}{|y|},$$

then the boundary condition for the Green's function is fulfilled, and consequently,

$$G(x,y)=\frac{1}{4\pi|x-y|}-\frac{R}{4\pi|y||x-y'|}.$$

6.4.21. Find the Green's function for the following domains:
1) the half-space $x_3 > l$, $n = 3$, $x = (x_1, x_2, x_3)$;
2) the half-plane $x_2 > l$, $n = 2$, $x = (x_1, x_2)$.

Solution of Problem 2. Choose the point y^1 symmetrically to y with respect to the line $x_2 = l$ (see fig. 6.4.2).

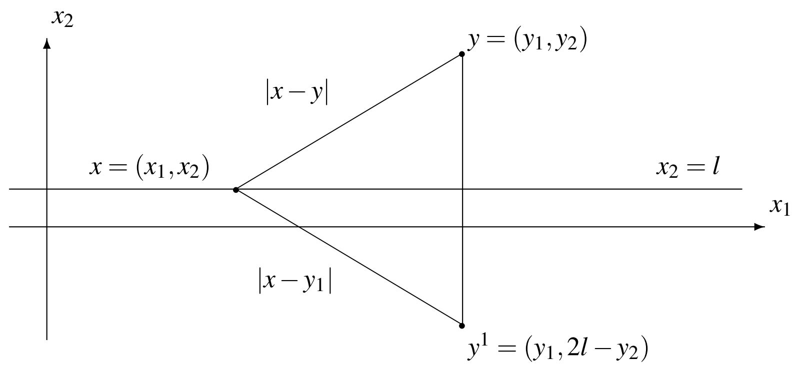

Figure 6.4.2.

We seek the function $v(x,y)$ in the form

$$v(x,y) = -\frac{1}{2\pi} \ln \frac{\alpha}{|x - y^1|};$$

where $y^1 = (y_1, 2l - y_2)$. If $x \in S$, then

$$\frac{1}{|x - y|} = \frac{1}{|x - y^1|}.$$

Hence, if $\alpha = 1$, then the boundary condition for the Green's function is fulfilled, and consequently,

$$G(x,y) = \frac{1}{2\pi} \ln \frac{1}{|x - y|} - \frac{1}{2\pi} \ln \frac{1}{|x - y^1|}.$$

We note that the construction of the point y^1 in Problems 6.4.20-6.4.21 is called the reflection of the point y with respect to the ball and the plane, respectively.

6.4.22. Construct the Green's function for the following domains in $\mathbf{R^3}$:
1) the dihedral angle $x_2 > 0$, $x_3 > 0$;
2) the dihedral angle $0 < \varphi < \frac{\pi}{n}$;
3) octant $x_1 > 0$, $x_2 > 0$, $x_3 > 0$;
4) the half-ball of radius R around the origin, $x_3 > 0$;
5) the quarter of the ball of radius R, $x_2 > 0$, $x_3 > 0$;
6) the part of the ball of radius R, with $x_1 > 0$, $x_2 > 0$, $x_3 > 0$;

7) the layer between the two planes $x_3 = 0$ and $x_3 = l$;
8) the half of the layer $0 < x_3 < l$, $x_1 > 0$.

Solution of Problem 6. Let us make reflections with respect to the parts of the boundary of the domain. First we reflect the point y with respect to the sphere, i.e. we construct the point

$$y' = \frac{R^2}{|y|^2}y.$$

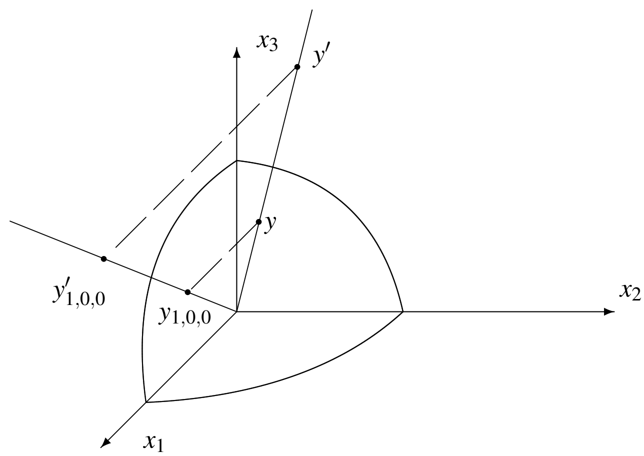

Figure 6.4.3.

The function

$$G_1(x,y) = \frac{1}{4\pi|x-y|} - \frac{R}{4\pi|y||x-y'|},$$

satisfies the boundary condition on the sphere, but $G_1(x,y)$ is not equal to zero on the other parts of the boundary. Let us now reflect the points y and y' with respect to the plane $x_1 = 0$. Let $y_{1,0,0}$ and $y'_{1,0,0}$ be the symmetrical points to y and y' respectively (see fig. 6.4.3). The function

$$G_2(x,y) = \frac{1}{4\pi|x-y|} - \frac{R}{4\pi|y||x-y'|} - \frac{1}{4\pi|x-y_{1,0,0}|} + \frac{R}{4\pi|y_{1,0,0}||x-y'_{1,0,0}|}.$$

is equal to zero on the sphere and on the plane $x_1 = 0$. Analogously we make reflections of the points y, y', $y_{1,0,0}$, $y'_{1,0,0}$ with respect to the plane $x_2 = 0$, and then we make reflections of the obtained points with respect to the plane $x_3 = 0$. Thus, the Green's function has the form

$$G(x,y) = \frac{1}{4\pi}\sum_{m,n,k=0}^{1}(-1)^{m+n+k}\Big(\frac{1}{|x-y_{m,n,k}|} - \frac{R}{|y||x-y'_{m,n,k}|}\Big),$$

where

$$y_{0,0,0} = y = (y_1,y_2,y_3),\ y_{m,n,k} = ((-1)^m y_1, (-1)^n y_2, (-1)^k y_3),$$

$$y'_{m,n,k} = y_{m,n,k} \frac{R^2}{|y_{m,n,k}|}.$$

6.4.23. Construct the Green's function for the following domains in $\mathbf{R^2}$:
1) the right angle $x_1 > 0,\ x_2 > 0$;
2) the angle $0 < \varphi < \frac{\pi}{n}$ (in the polar coordinates);
3) the half-disc of radius R around the origin, $x_2 > 0$;
4) the quarter of the disc of radius R, $x_1 > 0$, $x_2 > 0$;
5) the strip $0 < x_2 < \pi$;
6) the half-strip $0 < x_2 < l$, $x_1 > 0$.

Solution of Problem 2. Let us make reflections with respect to the parts of the boundary of the domain. It is convenient to use the polar coordinates $x = (\rho, \varphi)$, $y = (\mu, \psi)$. First we reflect the point y with respect to the line $\varphi = \frac{\pi}{n}$ (see fig. 6.4.4), i.e. we construct the point $\tilde{y}_1 = (\mu, \frac{2\pi}{n} - \psi)$.

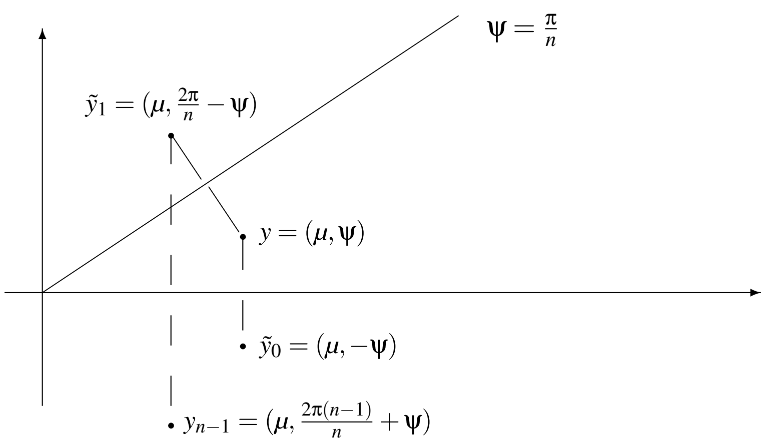

Figure 6.4.4.

The function

$$G_1(x,y) = \frac{1}{2\pi} \ln \frac{1}{|x-y|} - \frac{1}{2\pi} \ln \frac{1}{|x - \tilde{y}_1|},$$

is equal to zero on the line $\varphi = \frac{\pi}{n}$.

We reflect now the points y and $\tilde{y}_1$ with respect to the line $\varphi = 0$, and we obtain the function

$$G_2(x,y) = \frac{1}{2\pi} \ln \frac{1}{|x-y|} - \frac{1}{2\pi} \ln \frac{1}{|x - \tilde{y}_1|} - \frac{1}{2\pi} \ln \frac{1}{|x - \tilde{y}_0|} + \frac{1}{2\pi} \ln \frac{1}{|x - y_{n-1}|},$$

where

$$\tilde{y}_0 = (\mu, -\psi), \quad y_{n-1} = \left(\mu, -\frac{2\pi}{n} + \psi\right) = \left(\mu, \frac{2\pi(n-1)}{n} + \psi\right).$$

Furthermore, we again make a reflection with respect to the line $\varphi = \dfrac{\pi}{n}$. Obviously, we should reflect only the points $\tilde{y}_0$ and y_{n-1}, since the points y and $\tilde{y}_1$ lie symmetrically with respect to this line. We continue this procedure, and after the n-th reflection we obtain the Green's function of the form

$$G(x,y) = \frac{1}{2\pi}\sum_{k=0}^{n-1}\left(\ln\frac{1}{|x-y_k|} - \ln\frac{1}{|x-\tilde{y}_k|}\right),$$

where

$$y_k = \left(\mu, \frac{2\pi k}{n} + \psi\right), \quad \tilde{y}_k = \left(\mu, \frac{2\pi k}{n} - \psi\right), \quad y_0 = y, \quad k = \overline{0, n-1}.$$

The construction of the Green's function for a plane domain $D \subset \mathbf{R}^2$ connects with the problem of the conformal mapping of D onto the unit disc. Let $D \subset \mathbf{R}^2$ be a simply connected domain with a sufficiently smooth boundary S, $z = x_1 + ix_2 \in \overline{D}$, $\zeta = y_1 + iy_2 \in D$, and let $\omega(z,\zeta)$ be the function realizing the conformal mapping of D onto the unit disc, and $\omega(\zeta,\zeta) = 0$. Then the Green's function for D has the form

$$G(z,\zeta) = \frac{1}{2\pi}\ln\frac{1}{|\omega(z,\zeta)|}.$$

6.4.24. Construct the Green's function for the following domains in $\mathbf{R}^2$:
1) the half-plane $\Im z > 0$;
2) the quarter of the plane $0 < \arg z < \dfrac{\pi}{2}$;
3) the angle $0 < \arg z < \dfrac{\pi}{n}$;
4) the half-disc $|z| < R$, $\Im z > 0$;
5) the quarter of the disc $|z| < 1$, $0 < \arg z < \dfrac{\pi}{2}$;
6) the strip $0 < \Im z < \pi$;
7) the half-strip $0 < \Im z < \pi$, $\Re z > 0$.

Solution of Problem 6. Let us find the function realizing the conformal mapping of the strip into the unit disc. Let ζ be a fixed point of the strip. Using the function $z_1 = e^{\zeta}$ we map the strip $0 < \Im z < \pi$ into the upper half-plane. The point ζ is mapped into the point $\zeta_1 = e^{\zeta}$. Then we map the upper half-plane into the unit disc such that the point ζ_1 is mapped into the origin. This is made by the linear-fractional function

$$\omega(z_1) = e^{i\alpha}\frac{(z-\zeta_1)}{(z-\overline{\zeta_1})}.$$

Thus, the desired function has the form

$$\omega(z,\zeta) = e^{i\alpha}\frac{|e^z - e^{\zeta}|}{|e^z - \overline{e^{\zeta}}|},$$

and we construct the Green's function

$$G(z,\zeta)=\frac{1}{2\pi}\ln\frac{|e^z-\overline{e^\zeta}|}{|e^z-e^\zeta|}.$$

6.4.25. Using the Green's function find the solution of the Dirichlet problem

$$\Delta u(x_1,x_2,x_3)=0,\quad x\in D,$$

$$u|_S=g(x_1,x_2,x_3),$$

where $D=\{x:\ |x|<R\}$ is the ball of radius R.

Solution. The Green's function for the ball is constructed in Problem 4.20. Let us calculate the normal derivative on the sphere $\dfrac{\partial G(x,y)}{\partial n_y}$. The differentiation with respect to the normal coincides with the differentiation with respect to the radius $|y|=r$. Therefore, we introduce spherical coordinates. We get

$$|x-y|^2=|x|^2+|y|^2+2|x||y|\cos\gamma;$$

$$|x-y'|^2=|x|^2+|y'|^2+2|x||y'|\cos\gamma;\quad |y'|:=\frac{R^2}{|y|};$$

$$\frac{\partial G(x,y)}{\partial n_y}=\frac{1}{4\pi}\frac{\partial}{\partial r}\Big(\frac{1}{\sqrt{|x|^2+r^2-2|x|r\cos\gamma}}$$

$$-\frac{R}{\sqrt{|x|^2r^2+R^4-2|x|r\cos\gamma}}\Big)\Big|_{r=R}$$

$$=\frac{|x|^2-R^2}{4\pi R(R^2+|x|^2-2R|x|\cos\gamma)^{3/2}}=\frac{|x|^2-R^2}{4\pi R(|x-y|^3}\Big|_{y\in S}.$$

Thus, the solution of our Dirichlet problem is given by the formula (the Poisson formula):

$$u(x)=\frac{1}{4\pi R}\int_S\frac{R^2-|x|^2}{|x-y|^3}g(y)\,dy.$$

Analogously, for the disc ($n=2$) the Poisson formula has the form

$$u(x)=\frac{1}{2\pi R}\int_S\frac{R^2-|x|^2}{|x-y|^2}g(y)\,dy.$$

6.4.26. Using the Poisson formula solve Problems 6.4.13 and 6.4.17.

6.4.27. Using (6.4.3) solve Problems 6.4.15 and 6.4.16.

6.4.28. Find the solution of the Dirichlet problem

$$\Delta u(x_1,x_2,x_3)=-f(x_1,x_2,x_3),\quad x\in D,$$

$$u(x_1,x_2,x_3)|_S=g(x_1,x_2,x_3),$$

where D is the half-space $x_3 > 0$, and
1) $f(x) = 0, \quad g(x) = \cos x_1 \cos x_2$,
2) $f(x) = e^{-x_3} \sin x_1 \cos x_2, \quad g(x) = 0$.

6.4.29. Find the solution of the Dirichlet problem

$$\Delta u(x_1, x_2) = 0, \quad x \in D,$$

$$u(x_1, x_2)|_S = g(x_1, x_2),$$

where
1) D is the half-plane $x_2 > 0$, and $g(x_1, x_2) = \dfrac{x_1}{1+x_1^2}$;
2) D is the strip $0 < x_2 < \pi$, and $g(x_1, 0) = \cos x_1$, $g(x_1, \pi) = 0$.

3. Potentials and their applications

Let $D \subset \mathbf{R^n}$ $(n \ge 2)$ be a bounded domain with the smooth boundary S. Consider the Dirichlet problem for the Poisson equation:

$$\Delta V(x) = -f(x), \tag{6.4.4}$$

$$V(x)|_{x\in S} = g(x), \quad x = (x_1, \dots, x_n), \quad x \in D \in \mathbf{R^n}. \tag{6.4.5}$$

If a particular solution of equation (6.4.4) is known, then problem (6.4.4)-(6.4.5) can be reduced to the Dirichlet problem for the Laplace equation $\Delta V(x) = 0$, considered above. As a partial solution of (6.4.4) one can take the function

$$V(x) = \begin{cases} \dfrac{1}{\omega_n (n-2)} \displaystyle\int_D f(\tau) \frac{1}{|x-\tau|^{n-2}}\, d\tau, & n \ge 3, \\ \dfrac{1}{2\pi} \displaystyle\int_D f(\tau) \ln \frac{1}{|x-\tau|}\, d\tau, & n = 2, \end{cases} \tag{6.4.6}$$

where ω_n is the area of the unit sphere in $\mathbf{R^n}$, and

$$|x-\tau| = \sqrt{\sum_{i=1}^n (x_i - \tau_i)^2}.$$

The function $V(x)$ is called the *volume potential*, and the function $f(x)$ is called the *density*. For $n = 2$ the function $V(x)$ is called also the *logarithmic volume potential*. For example, for $n = 3$ the function $V(x)$ describes the potential of gravitational field created by the body D with the density f.

If $f(x) \in C^1(\overline{D})$, then the function $V(x)$ satisfies the equation

$$\Delta V(x) = \begin{cases} -f(x), & x \in D, \\ 0, & x \notin D. \end{cases}$$

6.4.30. Can a harmonic in D function be a volume potential with non-zero density?

6.4.31. Let $n=3$. Find the density $f(x_1,x_2,x_3)$ if the volume potential in D has the form

$$V(x_1,x_2,x_3) = (x_1^2+x_2^2+x_3^2)^2 - 1.$$

6.4.32. Show that the function

$$V(x_1,x_2) = \begin{cases} -\dfrac{1}{2}\ln|x|, & |x| \geq 1, \\ \dfrac{1}{4}(1-|x|^2), & |x| \leq 1 \quad \left(|x| = \sqrt{x_1^2+x_2^2}\right) \end{cases}$$

is a logarithmic volume potential for the disc $|x|<1$ with the density $f(x_1,x_2)=1$.

6.4.33. Show that the function

$$V(x_1,x_2,x_3) = \begin{cases} \dfrac{1}{3}, & |x| \geq 1, \\ \dfrac{1}{2}(1-\dfrac{|x|^2}{3}), & |x| \leq 1 \quad \left(|x| = \sqrt{x_1^2+x_2^2+x_3^2}\right) \end{cases}$$

is a volume potential for the disc $|x|<1$ with the density $f(x_1,x_2,x_3)=1$.

6.4.34. Let a domain $D \subset \mathbf{R}^2$ contains the square $-1 \leq x_1 \leq 1$, $-1 \leq x_2 \leq 1$, and let

$$V(x_1,x_2) = x_1^2 x_2^2$$

be the logarithmic volume potential for D. Determine the mass contained in the square.

6.4.35. The function

$$V(x_1,x_2) = \frac{\pi x_1}{4}\left(2-|x|^2\right), \quad |x| \leq 1$$

is the logarithmic volume potential for the disc $|x| \leq 1$. Find the density $f(x_1,x_2)$ and the value of the potential for $|x|>1$.

6.4.36. Find the volume potential $V(x)$ for the ball $|x| \leq R$ $(n=3)$ with the constant density f_0
1) by the direct calculation of the volume integral;
2) by solving the corresponding boundary value problem.

6.4.37. Find the volume potential in $\mathbf{R}^3$ for the body distributed in
1) the spherical layer $a \leq |x| \leq b$ with the constant density f_0;
2) the ball $|x| \leq a$ with the constant density f_1, and the spherical layer $a<b<|x|<c$ with the constant density f_2. .

Consider the function

$$W(x) = \begin{cases} -\dfrac{1}{\omega_n(n-2)}\displaystyle\int_S V(\tau)\frac{\partial}{\partial n_\tau}\left(\frac{1}{|x-\tau|^{n-2}}\right) ds(\tau), & n \geq 3, \\ -\dfrac{1}{2\pi}\displaystyle\int_S V(\tau)\frac{\partial}{\partial n_\tau}\left(\ln\frac{1}{|x-\tau|}\right) ds(\tau), & n=2, \end{cases} \tag{6.4.7}$$

where n_τ is the outer normal to S at the point $\tau \in S$. The function $W(x)$ is called the *double-layer potential* with the density V. For $n = 2$ the function $W(x)$ is also called the *logarithmic double-layer potential*.

Physical sense: $W(x)$ is the potential of the field created by the dipole distribution on S with the density V.

It is convenient to calculate the integral in (6.4.7) by the formula

$$W(x) = \begin{cases} \dfrac{1}{\omega_n}\displaystyle\int_S V(\tau)\frac{\cos\varphi}{|x-\tau|^{n-1}}\,ds(\tau), & n \ge 3, \\ \dfrac{1}{2\pi}\displaystyle\int_S V(\tau)\frac{\cos\varphi}{|x-\tau|}\,ds(\tau), & n = 2, \end{cases}$$

where φ is the angle between n_τ and the vector directed from the point x to the point τ.

If $V(\tau)$ is continuous on S, then $W(x)$ is harmonic in D and $D_1 := \mathbf{R^n} \setminus \overline{D}$. Denote

$$W^+(x) = \lim_{\substack{x' \to x \\ x' \in D}} W(x'), \quad W^-(x) = \lim_{\substack{x' \to x \\ x' \in D_1}} W(x').$$

The function $W(x)$ has a jump on S:

$$W^+(x) = W(x) + \frac{1}{2}V(x),\ W^-(x) = W(x) - \frac{1}{2}V(x), \quad x \in S. \tag{6.4.8}$$

6.4.38. Let $D = \{x \in \mathbf{R^2} : |x| < r\}$, and let S be the boundary of D. Find the solution of the Dirichlet problem

$$\Delta W(x) = 0, \quad x \in D, \tag{6.4.9}$$

$$W(x)|_S = g(x). \tag{6.4.10}$$

Solution. We seek the desired function $W(x)$ in the form of a logarithmic double-layer potential:

$$W(x) = \frac{1}{2\pi}\int_S V(\tau)\frac{\cos\varphi}{|x-\tau|}\,ds(\tau). \tag{6.4.11}$$

It follows from (6.4.10) that $W^+(x) = g(x)$ for $x \in S$. Using (6.4.8) we obtain

$$\frac{1}{2}V(x) + \frac{1}{2\pi}\int_S V(\tau)\frac{\cos\varphi}{|x-\tau|}\,ds(\tau) = g(x).$$

Since

$$\frac{\cos\varphi}{|x-\tau|} = \frac{1}{2r},$$

we have

$$V(x) + \frac{1}{2\pi r}\int_S V(\tau)\,ds(\tau) = 2g(x). \tag{6.4.12}$$

Then

$$V(x) = 2g(x) + A \quad (A - const). \tag{6.4.13}$$

Using (6.4.12) we calculate

$$A = -\frac{1}{2\pi r}\int_S g(\tau)\,ds(\tau). \tag{6.4.14}$$

Relations (6.4.11), (6.4.13) and (6.4.14) give us the solution of the problem (6.4.9)-(6.4.10).

Using (6.4.11), (6.4.13) and (6.4.14) one can obtain the solution of the problem (6.4.9)-(6.4.10) in the form of a Poisson integral. Indeed, for $x \in D$ we have

$$W(x) = \frac{1}{2\pi}\int_S \frac{\cos\varphi}{|x-\tau|}\Big(2g(\tau) - \frac{1}{2\pi r}\int_S g(\xi)ds(\xi)\Big)\,ds(\tau)$$

$$= \frac{1}{\pi}\int_S \frac{\cos\varphi}{|x-\tau|}g(\tau)ds(\tau) - \Big(\frac{1}{4\pi^2 r}\int_S g(\xi)\,ds(\xi)\Big)\int_S \frac{\cos\varphi}{|x-\tau|}\,ds(\tau)$$

$$= \frac{1}{\pi}\int_S \frac{\cos\varphi}{|x-\tau|}g(\tau)\,ds(\tau) - \Big(\frac{1}{4\pi^2 r}\int_S g(\xi)\,ds(\xi)\Big)\cdot 2\pi$$

$$= \frac{1}{\pi}\int_S \Big(\frac{\cos\varphi}{|x-\tau|} - \frac{1}{2r}\Big)g(\tau)\,ds(\tau). \qquad (6.4.15)$$

Here we use the relation

$$W(x)|_{V\equiv 1} = \frac{1}{2\pi}\int_S \frac{\cos\varphi}{|x-\tau|}\,ds(\tau) = 1, \quad x \in D.$$

Applying the cosine theorem we calculate

$$\frac{\cos\varphi}{|x-\tau|} - \frac{1}{2r} = \frac{2r\cos\varphi - |x-\tau|}{2r|x-\tau|} = \frac{2r\cos\varphi|x-\tau| - |x-\tau|^2}{2r|x-\tau|^2}$$

$$= \frac{r^2-\rho_0^2}{2r(r^2+\rho_0^2 - 2r\rho_0\cos(\theta-\theta_0))}, \qquad (6.4.16)$$

where (ρ_0, θ_0) are the polar coordinates of the point (x_1, x_2), and θ is the polar angle of the point $\tau \in S$. Substituting (6.4.16) into (6.4.15) and using the polar coordinates we obtain the Poisson integral:

$$W(\rho_0,\theta_0) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(r^2-\rho_0^2)g(\theta)d\theta}{r^2+\rho_0^2-2r\rho_0\cos(\theta-\theta_0)}.$$

6.4.39. Using the double-layer potential solve the Dirichlet problem:

$$\Delta W(x) = 0, \quad x \in D, \quad W(x)|_S = g(x)$$

for the following domains
1) $D = \{x \in \mathbf{R^2} : |x| > r\}$;
2) $D = \{x \in \mathbf{R^2} : x_2 > 0\}$;
3) $D = \{x \in \mathbf{R^3} : x_3 > 0\}$.

6.4.40. Find the logarithmic double-layer potential for the circle $|x| = 1$
1) with the constant density V_0,
2) with the density $V(x_1,x_2) = x_1$.

The function

$$u(x)=\begin{cases}\dfrac{1}{\omega_n(n-2)}\displaystyle\int_S \mu(\tau)\frac{1}{|x-\tau|^{n-2}}\,ds(\tau), & n\ge 3,\\ \dfrac{1}{2\pi}\displaystyle\int_S \mu(\tau)\ln\frac{1}{|x-\tau|}\,ds(\tau), & n=2\end{cases}$$

is called the *single-layer potential*. For $n=2$ the function $u(x)$ is also called the *logarithmic single-layer potential*.

Physical sense: $u(x)$ is the potential of the field created by charges distributed on S with the density μ.

If the function $\mu(\tau)$ is continuous on S, then $u(x)$ is continuous everywhere in $\mathbf{R^n}$ and harmonic for $x\in\mathbf{R^n}\setminus S$. Denote

$$\left(\frac{\partial u}{\partial n_x}\right)^+=\lim_{\substack{x'\to x\\ x'\in D}}\frac{\partial u(x')}{\partial n_x},\quad \left(\frac{\partial u}{\partial n_x}\right)^-=\lim_{\substack{x'\to x\\ x'\in D_1}}\frac{\partial u(x')}{\partial n_x},$$

where $\dfrac{\partial u(x)}{\partial n_x}$ is the direct value of the normal derivative on S. Then

$$\left(\frac{\partial u}{\partial n_x}\right)^+=\frac{\partial u(x)}{\partial n_x}+\frac{1}{2}\mu(x),\quad x\in S, \tag{6.4.17}$$

$$\left(\frac{\partial u}{\partial n_x}\right)^-=\frac{\partial u(x)}{\partial n_x}-\frac{1}{2}\mu(x),\quad x\in S. \tag{6.4.18}$$

The single-layer potential is used for the solution of the Neumann problem.

6.4.41. Find the behavior of the single-layer and double-layer potentials as $|x|\to\infty$, $n=2,3$.

6.4.42. Find the single-layer potential $u(x)\quad (x\in\mathbf{R^3})$ for the sphere $|x|=1$ with the constant density μ_0
1) by direct calculation of the integral;
2) by solving the boundary value problem for $u(x)$.

6.4.43. Find the logarithmic single-layer potential $u(x)$ for the circle $|x|=R$ with the constant density $\mu_(x)=1$.

6.4.44. The function

$$u(x_1,x_2)=-\frac{x_2}{2|x|},\quad |x|>1,$$

is the logarithmic single-layer potential for the circle $|x|=1$. Find $u(x_1,x_2)$ for $|x|<1$.

6.4.45. The function

$$u(x_1,x_2)=\frac{x_1}{|x|^2}\left(1+\frac{2x_2}{|x|^2}\right),\quad |x|\le 1$$

is the logarithmic single-layer potential for the circle $|x|=1$. Find the density $\mu(x_1,x_2)$.

6.4.46. Using the logarithmic single-layer potential solve the Neumann problem:

$$\Delta u(x) = 0, \quad x \in D := \{x \in \mathbf{R}^2 : \ |x| < r\},$$
$$\left.\frac{\partial u}{\partial n}\right|_{x\in S} = g(x).$$

6.5. Answers and Hints

6.1.1.

2. $u_{\eta\eta} + 21u_\xi + 7u_\eta - 3 = 0,$

   $\xi = 3x + y, \quad \eta = x;$

3. $u_{\xi\eta} + u_\xi + 3u_\eta + \eta + 1 = 0,$

   $\xi = 2x + y, \quad \eta = x;$

4. $u_{\xi\eta} + 6u_\xi - 18u_\eta - 10u + 2\xi = 0,$

   $\xi = y, \quad \eta = x - 3y;$

5. $u_{\xi\xi} + u_{\eta\eta} + \frac{3}{16}u_\xi - \frac{10}{16}u_\eta + \frac{9}{16}u + \frac{1}{64}(6\xi + \eta - 8) = 0,$

   $\xi = y, \quad \eta = 4x - 2y;$

7. $u_{\xi\eta} + \frac{1}{6(\xi+\eta)}(u_\xi + u_\eta) = 0,$

   $\xi = \frac{2}{3}x^{3/2} + y, \quad \eta = \frac{2}{3}x^{3/2} - y \quad (x > 0);$

   $u_{\xi\xi} + u_{\eta\eta} + \frac{1}{3\xi}u_\xi = 0,$

   $\xi = \frac{2}{3}(-x)^{3/2}, \ \eta = y \quad (x < 0);$

8. $u_{\xi\xi} = 0,$

   $\xi = x + y, \quad \eta = x - y \quad (y > 0);$

   $u_{\xi\eta} = 0,$

   $\xi = (1+\sqrt{2})x + y, \quad \eta = (1-\sqrt{2})x + y \quad (y < 0);$

9. $u_{\xi\xi} - u_{\eta\eta} - \frac{1}{\xi}u_\xi + \frac{1}{\eta}u_\eta = 0,$

   $\xi = \sqrt{|x|}, \ \eta = \sqrt{|y|} \quad (xy > 0);$

   $u_{\xi\xi} + u_{\eta\eta} - \frac{1}{\xi}u_\xi - \frac{1}{\eta}u_\eta = 0,$

   $\xi = \sqrt{|x|}, \ \eta = \sqrt{|y|} \quad (xy < 0);$

10. $u_{\xi\xi} - u_{\eta\eta} + \frac{1}{3\xi}u_\xi - \frac{1}{3\eta}u_\eta = 0,$

$\xi = |x|^{3/2}, \quad \eta = |y|^{3/2} \quad (xy > 0);$

$u_{\xi\xi} + u_{\eta\eta} + \frac{1}{3\xi}u_\xi + \frac{1}{3\eta}u_\eta = 0,$

$\xi = |x|^{3/2}, \quad \eta = |y|^{3/2} \quad (xy < 0);$

11. $u_{\xi\eta} - \frac{1}{2\xi}u_\eta = 0,$

$\xi = xy, \quad \eta = \frac{y}{x} \quad (xy \neq 0);$

12. $u_{\xi\xi} + u_{\eta\eta} - u_\xi - u_\eta = 0,$

$\xi = \ln|x|, \quad \eta = \ln|y| \quad (xy \neq 0);$

13. $u_{\xi\xi} + u_{\eta\eta} + \frac{1}{2\xi}u_\xi + \frac{1}{2\eta}u_\eta = 0,$

$\xi = y^2, \quad \eta = x^2 \quad (xy \neq 0);$

14. $u_{\xi\eta} + \frac{1}{2(\eta^2 - \xi^2)}(\eta u_\xi - \xi u_\eta) = 0,$

$\xi = y;$

15. $u_{\xi\xi} + u_{\eta\eta} = 0,$

$\xi = y, \quad \eta = \arctan x;$

16. $u_{\xi\xi} + u_{\eta\eta} - 2u = 0,$

$\xi = \ln(x + \sqrt{1+x^2}), \quad \eta = \ln(y + \sqrt{1+y^2});$

17. $u_{\xi\eta} + 2\frac{\xi^2}{\eta^2}u_\xi + \frac{1}{\eta}e^\xi = 0,$

$\xi = \frac{y}{x}, \quad \eta = y;$

18. $u_{\eta\eta} - \frac{2\xi}{\xi^2 + \eta^2}u_\xi = 0,$

$\xi = y \tan\frac{x}{2}, \quad \eta = y;$

19. $u_{\xi\eta} - \frac{1}{2(\xi - \eta)}(u_\xi - u_\eta) + \frac{1}{4(\xi + \eta)}(u_\xi + u_\eta) = 0,$

$\xi = y^2 + e^{-x}, \quad \eta = y^2 - e^x, \quad (y \neq 0);$

20. $u_{\xi\eta} + \frac{1}{4\eta}u_\xi - \frac{1}{\xi}u_\eta + u = 0,$

$\xi = xy, \quad \eta = \dfrac{x^3}{y}, \quad (xy \neq 0);$

21. $u_{\eta\eta} - 2u_{\xi} = 0,$

$\xi = 2x - y^2, \quad \eta = y;$

22. $u_{\eta\eta} + \dfrac{2\eta^2}{\xi - \eta^2} u_{\xi} - \dfrac{1}{\eta} u_{\eta} = 0,$

$\xi = x^2 + y^2, \quad \eta = x, \quad (x \neq 0);$

23. $u_{\xi\eta} = 0,$

$\xi = x + y - \cos x, \quad \eta = -x + y - \cos x;$

24. $u_{\xi\eta} + \dfrac{1}{2(\xi - \eta)}(u_{\xi} - u_{\eta}) = 0,$

$\xi = y + 2\sqrt{x} - x, \quad \eta = y - x - 2\sqrt{x}, \quad (x > 0);$

$u_{\xi\xi} + u_{\eta\eta} - \dfrac{1}{\eta} u_{\eta} = 0,$

$\xi = y - x, \quad \eta = 2\sqrt{-x}, \quad (x < 0);$

6.1.2.

1. $v_{\xi\xi} + v_{\eta\eta} - 11v = 0,$

$\xi = y - \dfrac{x}{2}, \quad \eta = \dfrac{x}{2}, \quad \lambda = -3, \quad \mu = 2;$

2. $v_{\xi\eta} - \dfrac{4}{49} e^{\frac{2}{7}\xi} = 0,$

$\xi = x - 3y, \quad \eta = 2x + y, \quad \lambda = -\dfrac{2}{7}, \quad \mu = 0;$

3. $v_{\eta\eta} + v_{\xi} = 0,$

$\xi = y - x, \quad \eta = x, \quad \lambda = -\dfrac{3}{4}, \quad \mu = -\dfrac{1}{2};$

4. $v_{\xi\eta} - \dfrac{26}{3} v + \dfrac{2(\xi + \eta)}{27} e^{2\xi + 4\eta} = 0,$

$\xi = 2x - y, \quad \eta = x + y, \quad \lambda = -2, \quad \mu = 4;$

5. $v_{\xi\eta} - \dfrac{7}{48} v = 0,$

$\xi = y + (\sqrt{3} - 2)x, \quad \eta = y - (\sqrt{3} - 2)x, \quad \lambda = \mu = \dfrac{1}{4};$

6. $v_{\eta\eta} - \dfrac{11}{4} v_{\xi} = 0,$

$\xi = x + 2y, \quad \eta = x, \quad \lambda = -\dfrac{11}{16}, \quad \mu = \dfrac{5}{8};$

7. $v_{\xi\xi} + v_{\eta\eta} - \frac{297}{4} v = 0,$

$\xi = x, \quad \eta = 3x + y, \quad \lambda = -\frac{7}{2}, \quad \mu = -8;$

8. $v_{\xi\eta} - \frac{87}{64} v = 0,$

$\xi = x - y, \quad \eta = 3x + y, \quad \lambda = -\frac{5}{8}, \quad \mu = -\frac{3}{8};$

9. $v_{\xi\eta} - \frac{5}{4} v = 0,$

$\xi = x + y, \quad \eta = 3x - y, \quad \lambda = 2, \quad \mu = \frac{1}{2};$

10. $v_{\xi\xi} + v_{\eta\eta} + \frac{11}{4} v + 2\eta e^{\frac{\xi}{2}} = 0,$

$\xi = 2x - y, \quad \eta = x, \quad \lambda = -\frac{1}{2}, \quad \mu = 0;$

11. $v_{\xi\xi} + v_{\eta\eta} - \frac{9}{4} v + (\eta - \xi) e^{-\xi + \frac{\eta}{2}} = 0,$

$\xi = 2x - y, \quad \eta = 3x, \quad \lambda = 1, \quad \mu = -\frac{1}{2};$

12. $v_{\eta\eta} - 6 v_{\xi} = 0,$

$\xi = x + y, \quad \eta = x, \quad \lambda = -\frac{37}{24}, \quad \mu = -\frac{1}{2};$

13. $v_{\xi\eta} - 2v = 0, \quad \xi = x - y, \quad \eta = x + y, \quad \lambda = -1, \quad \mu = 0;$

14. $v_{\eta\eta} - \frac{5}{9} v_{\xi} + \frac{1}{9} (2\eta - \xi) e^{-\frac{16}{45}\xi - \frac{4}{9}\eta} = 0,$

$\xi = x + 3y, \quad \eta = x, \quad \lambda = -\frac{16}{45}, \quad \mu = \frac{4}{9};$

15. $v_{\xi\eta} - 26v = 0,$

$\xi = 2x - y, \quad \eta = x, \quad \lambda = -2, \quad \mu = -12;$

16. $v_{\xi\eta} + 16v + 8(\xi - \eta) e^{\xi + \eta} = 0,$

$\xi = y - x, \quad \eta = y, \quad \lambda = \mu = -2;$

17. $v_{\xi\xi} + v_{\eta} = 0,$

$\xi = 2x - y, \quad \eta = x + y, \quad \lambda = 1, \quad \mu = 2.$

6.1.3.

1. $u_{\xi\xi}+u_{\eta\eta}+u_{\gamma\gamma}=0,$

   $\xi=y,\quad \eta=x-y,\quad \gamma=y-\frac{1}{2}x+\frac{1}{2}z;$

2. $u_{\xi\xi}+u_{\eta\eta}+u_{\gamma\gamma}=0,$

   $\xi=z,\quad \eta=y-x,\quad \gamma=x-2y+2z;$

3. $u_{\xi\xi}-u_{\eta\eta}+7u=0,$

   $\xi=y+z,\quad \eta=-y-2z,\quad \gamma=x-z;$

4. $u_{\xi\xi}+5u=0,$

   $\xi=y,\quad \eta=x-2y,\quad \gamma=-y+z;$

5. $u_{\xi\xi}-u_{\eta\eta}+u_{\gamma\gamma}+u_{\eta}=0,$

   $\xi=\frac{1}{2}x,\quad \eta=\frac{1}{2}x+y,\quad \gamma=-\frac{1}{2}x-y+z;$

6. $u_{\xi\xi}-2u_{\xi}=0,$

   $\xi=z,\quad \eta=y-2z,\quad \gamma=x-3z;$

7. $u_{\xi\xi}+u_{\eta\eta}+u_{\gamma\gamma}+u_{\tau\tau}=0,$

   $\xi=t,\quad \eta=y-t,\quad \gamma=t-y+z,\quad \tau=x-2y+z+2t;$

8. $u_{\xi\xi}+u_{\eta\eta}+u_{\gamma\gamma}-u_{\tau\tau}=0,$

   $\xi=x+y,\quad \eta=y+z-t,\quad \gamma=t,\quad \tau=y-x;$

9. $u_{\xi\xi}-u_{\eta\eta}+u_{\gamma\gamma}=0,$

   $\xi=x,\quad \eta=y-x,\quad \gamma=2x-y+z,\quad \tau=x+z+t.$

6.1.4.

1. The change of variables

   $$\xi=x+y,\ \eta=3x+2y$$

   reduces the equation to the canonical form $u_{\xi\eta}=0$ with the general solution $u=f(\xi)+g(\eta)$, hence

   $$u(x,y)=f(x+y)+g(3x+2y),$$

   where f and g are arbitrary twice continuously differentiable functions;

2. $u(x,y)=f(y-x)+\exp((x-y)/2)g(y-2x);$

3. $u(x,y)=f(x+3y)+g(3x+y)\exp((7x+y)/16);$

4. The change of variables
$$\xi = y - 3x,\ \eta = 3y - x$$
reduces the equation to the canonical form
$$32u_{\xi\eta} + u_\xi - u_\eta - \frac{v}{32} - (3\xi - \eta)\exp\left(\frac{\xi-\eta}{32}\right) = 0.$$
The replacement
$$u(\xi,\eta) = \exp\left(\frac{\xi-\eta}{32}\right) v(\xi,\eta)$$
yields
$$32w_{\xi\eta} - 3\xi + \eta = 0.$$
Integrating the equation we get
$$u(x,y) = \left(f(y-3x) + g(3y-x) - \frac{1}{8}x(y-3x)(3y-x)\right)$$
$$\times \exp\left(-\frac{(x+y)}{16}\right).$$

5. $u(x,y) = 2e^x + e^{(x+2y)/2}(f(x) + g(x+2y))$;

6. $u(x,y) = e^{x+y/2}((2x+y)e^{4x+y} = f(2x+y) + g(4x+y))$;

7. $u(x,y) = f(y+2x+\sin x) + e^{-(y+2x+\sin x)/4} g(y-2x+\sin x)$;

8. $u(x,y) = e^y(e^{2y} - e^{2x}) + f(e^y + e^x) + g(e^y - e^x)$;

9. Denote $v = u_y$. By differentiation we get the following equation with respect to $v(x,y)$:
$$v_{xy} + yv_y = 0.$$
The general solution of this equation is
$$v(x,y) = g(x) + \int_0^y f(\eta) e^{-\eta x}\, d\eta,$$
where f and g are arbitrary smooth functions. Hence
$$u(x,y) = yg(x) + g'(x)\int_0^y (y-\eta) f(\eta) e^{-\eta x}\, d\eta.$$

6.2.1.

1. $u(x,t) = \sin x \cos t$;

2. $u(x,t) = \dfrac{A}{a}\sin x \sin at$;

3. $u(x,t) = \frac{1}{2}[\varphi(x+a(t-t_0)) + \varphi(x-a(t-t_0))]$

$+\frac{1}{2a}\int\limits_{x-a(t-t_0)}^{x+a(t-t_0)} \psi(\xi)d\xi;$

4. $u(x,t) = (x+t)^2 + 2xt.$

In problems 6.2.2-6.2.5 reduce the equations to the canonical forms.

6.2.2. $u(x,y) = 3x^2 + y^2;$

6.2.3. $u(x,y) = \varphi_0(x - \frac{2}{3}y^3) + \frac{1}{2}\int_{x-\frac{2}{3}y^3}^{x+2y} \varphi_1(x)dx;$

6.2.4.

$$u(x,y) = \frac{\varphi_0(x-\sin x+y) + \varphi_0(x+\sin x-y)}{2} + \frac{1}{2}\int\limits_{x+\sin x-y}^{x-\sin x+y} \varphi_1(z)dz;$$

6.2.5. $u(x,y) = \frac{1}{2}(f(x-2\sqrt{-y}) + f(x+2\sqrt{y})); \quad (y<0).$

6.2.6.

1. $u(x,t) = (x+2t)^2;$
2. $u(x,t) = x + \frac{xt^3}{6} + \sin x \sin t;$
3. $u(x,t) = \sin x;$
4. $u(x,t) = (1-\cos t)\sin x;$
5. $u(x,t) = x(t-\sin t) + \sin(x+t);$
6. $u(x,t) = \sin(x+t) + xt - (1-\operatorname{ch} t)e^x;$
7. $u(x,t) = t + e^{-t} - 1 + \sin(x+t).$

6.2.7.

1. Extend $\varphi(x)$ and $\psi(x)$ on the half-line $-\infty < x < 0$ as even functions.
2. $u(x,t) = \psi(x+at) - \psi(x-at)$, where

$$\psi(z) = \frac{1}{2a}\int_{-2l}^{z} \varphi(\xi)\,d\xi,$$

$$\varphi(z) = \begin{cases} 0, & z \in [(-\infty;-2l)\bigcup(-l;l)\bigcup(2l;\infty)], \\ v_0, & z \in [(-2l;-l)\bigcup(l;2l)]; \end{cases}$$

3. The boundary condition causes a wave propagating from $x=0$ in the direction of the x-axis; hence we seek a solution of the problem in the form $u(x,t)=f(x-at)$;

$$u(x,t)=\begin{cases} 0 & \text{for} \quad t<\dfrac{x}{a}, \\ A\sin\omega\left(t-\dfrac{x}{a}\right) & \text{for} \quad t>\dfrac{x}{a}. \end{cases}$$

6.2.9. We use the reflection method and seek the solution in the form (2.1.13), where Φ and Ψ should be constructed. The initial conditions define these functions in the interval $(0,l)$. We extend Φ and Ψ as odd functions with respect to the points $x=0$ and $x=l$:

$$\Phi(x)=-\Phi(-x), \quad \Phi(x)=-\Phi(2l-x),$$

$$\Psi(x)=-\Psi(-x), \quad \Psi(x)=-\Psi(2l-x),$$

i.e.

$$\Phi(-x)=\Phi(2l-x), \quad \Psi(-x)=\Psi(2l-x),$$

hence $\Phi(x)$ and $\Psi(x)$ are $2l$ - periodic functions.

6.2.10.

1. $u(x,t)=A\sin\dfrac{\pi x}{l}\cos\dfrac{\pi at}{l} \quad (0<x<l,\, t>0);$

2. $u(x,t)=\dfrac{\varphi(x-at)+\varphi(x+at)}{2} \quad (0<x<l,\, t>0);$

where

$$\varphi(z)=\begin{cases} Az & \text{for} \quad -l<z<l, \\ A(2l-z) & \text{for} \quad l<z<3l, \end{cases}$$

$$\varphi(z)=\varphi(z+4l), \quad -\infty<z<\infty.$$

6.2.11.

2. $\lambda_n=\left(\dfrac{n\pi}{l}\right)^2, \quad y_n(x)=\cos\dfrac{n\pi}{l}x, \quad n=0,1,2,\ldots;$

3. $\lambda_n=\left(\dfrac{(2n+1)\pi}{2l}\right)^2, \quad y_n(x)=\sin\dfrac{(2n+1)\pi}{2l}x, \quad n=0,1,2,\ldots;$

4. $\lambda_n=\left(\dfrac{(2n+1)\pi}{2l}\right)^2, \quad y_n(x)=\cos\dfrac{(2n+1)\pi}{2l}x, \quad n=0,1,2,\ldots;$

5. $y_n(x)=\cos\rho_n x,$ where ρ_n are roots of the equation $\rho\sin\rho l-h\cos\rho l=0.$

6.2.12.

1. $\lambda_n=\left(\dfrac{n\pi}{l}\right)^2+\gamma, \quad y_n(x)=\sin\dfrac{n\pi}{l}x, \quad n=1,2,\ldots;$

2. $\lambda_n = \left(\frac{n\pi}{l}\right)^2 + \gamma, \quad y_n(x) = \cos\frac{n\pi}{l}x, \quad n = 0,1,2,\ldots;$

3. $\lambda_n = \left(\frac{(2n+1)\pi}{2l}\right)^2 + \gamma, \quad y_n(x) = \sin\frac{(2n+1)\pi}{2l}x, \quad n = 0,1,2,\ldots;$

4. $\lambda_n = \left(\frac{n\pi}{l}\right)^2 + \gamma + \frac{\eta^2}{4}, \quad y_n(x) = e^{\frac{\eta x}{2}}\sin\frac{n\pi}{l}x, \quad n = 1,2,\ldots.$

6.2.13. Apply the method of separation of variables to the equation $u_{tt} = a^2u_{xx}$ with the boundary conditions $u(0,t) = u(l,t) = 0$. The corresponding Sturm-Liouville problem is solved in Problem 6.2.11,1.

6.2.14. Apply the method of separation of variables to the equation $u_{tt} = a^2u_{xx}$ with the boundary conditions $u_x(0,t) = u_x(l,t) = 0$. The corresponding Sturm-Liouville problem is solved in Problem 6.2.11:2.

6.2.15. Solve the equation $u_{tt} = a^2u_{xx}$ with the given initial and boundary conditions. The corresponding Sturm-Liouville problems are solved in Problems 6.2.11: 3-5.

6.2.16.
6.

$$u(x,t) = \sum_{n=1}^{\infty}\left(lJ_1^2(\mu_n)\right)^{-1} J_0\left(\mu_n\sqrt{\frac{x}{l}}\right)\cos\frac{\mu_n at}{2\sqrt{l}}\int_0^l \varphi(\xi)J_0\left(\mu_n\sqrt{\frac{\xi}{l}}\right)d\xi,$$

where J_p are the Bessel function, and $\mu_1,\mu_2,\mu_3,\ldots$ are the positive roots of the equation $J_0(\mu) = 0$.

6.2.17. The problem is reduced to the solution of the equation $u_{tt} = a^2u_{xx} + A\sin\omega t$ with the boundary conditions $u(0,t) = u(l,t) = 0$ and with zero initial conditions. A resonance appears for $\omega = \frac{an\pi}{l}$, $n = 1,2,\ldots.$

6.2.18. Solve the equation $u_{tt} = a^2u_{xx} + f(x,t)$ with the initial conditions $u(x,0) = u_t(x,0) = 0$ and with the given boundary conditions.

6.2.19. The problem is reduced to the solution of the equation $u_{tt} = a^2u_{xx}$ with the boundary conditions $u(0,t) = 0$, $u(l,t) = A\sin\omega t$ and with zero initial conditions.

6.2.20. Solve the equation $u_{tt} = a^2u_{xx} + f(x,t)$ with the given conditions.

6.2.21. Solve the problem $u_{tt} + 2\alpha u_t = a^2u_{xx} \quad (\alpha > 0)$, $u(x,0) = \varphi(x)$, $u_t(x,0) = \psi(x)$, $u(0,t) = u(l,t) = 0$.

6.2.22.
3. By the replacement

$$u(x,t) = e^{-\frac{t}{2}}(v(x,t) + (1-x)t),$$

Problem 3 is reduced to the following problem for the function $v(x,t)$:

$$v_{tt} = v_{xx} + \frac{1}{4}v + (x-1)e^{-\frac{t}{2}},$$

$$v(0,t) = v_t(0,t) = 0, \quad v(x,t) = v(1,t) = 0.$$

We seek the solution in the form

$$v(x,t) = \sum_{n=1}^{\infty} T_n(t) y_n(x),$$

where the functions $y_n(x)$ are the eigenfunctions of problem:

$$y'' + \left(\lambda + \frac{1}{4}\right) y = 0, \; y(0) = y(l) = 0,$$

and the functions $T_n(t)$ are the solutions of the Cauchy problem

$$T_n'' + \lambda_n T_n = \frac{1}{\pi n} e^{-\frac{t}{2}}, \quad T_n(0) = T_n'(0) = 0.$$

Solving the spectral problem we get:

$$y_n(x) = \sin \pi n x, \quad \lambda_n = (\pi n)^2 - \frac{1}{4}, \quad n \geq 1.$$

Solving the Cauchy problem we obtain

$$T_n(t) = \frac{1}{\pi n \sqrt{\lambda_n}(1 - 4\lambda_n)} \Big(2 \sin \sqrt{\lambda_n} t + 4\sqrt{\lambda_n} e^{-\frac{t}{2}} - 4\sqrt{\lambda_n} \cos \sqrt{\lambda_n} t \Big).$$

Then the solution of Problem 3 has the form:

$$u(x,t) = e^{-\frac{t}{2}} \Big(\sum_{n=1}^{\infty} \frac{1}{\pi n \sqrt{\lambda_n}(1 - 4\lambda_n)} \Big(2 \sin \sqrt{\lambda_n} t$$

$$+ 4\sqrt{\lambda_n} e^{-\frac{t}{2}} - 4\sqrt{\lambda_n} \cos \sqrt{\lambda_n} t \Big) \sin \pi n x + (1 - x) t \Big).$$

6.2.23.

2. $u = \sin x \cos t + x^2 t + \dfrac{t^3}{3} - t^2;$

3. $u = x \operatorname{ch} t + (t+1) \operatorname{sh} t - \dfrac{t^2}{2} e^x;$

   Hint. Make the substitution $u(x,t) = e^{-x} v(x,t)$.

4. $u = e^t \left(x - xt - \dfrac{t^2}{2} \right).$

   Hint. Make the substitution $u(x,t) = e^{-x} v(x,t)$.

6.2.24.

$$u(x,t) = \frac{\varphi(x-at)+\varphi(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \psi(z)dz$$
$$+\frac{1}{2a}\int_0^t d\tau \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(z,\tau)\,dz.$$

6.2.25.

$$u(x,y) = \frac{1}{2}\varphi(x,y) + \frac{y}{2}\varphi(\frac{x}{y}) + \frac{\sqrt{xy}}{y}\int_{xy}^{\frac{x}{y}} \frac{\varphi(z)}{z^{3/2}}dz - \frac{\sqrt{xy}}{2}\int_{xy}^{\frac{x}{y}} \frac{\psi(z)}{z^{3/2}}dz.$$

6.3.1.

1. $(1+t)^{-1/2}\exp\left(\frac{2x-x^2+t}{t+1}\right)$;
2. $(1+t)^{-1/2}\sin\frac{x}{1+t}\exp\left(-\frac{4x^2+t}{4(1+t)}\right)$;
3. $x(1+4t)^{-3/2}\exp\left(-\frac{x^2}{1+4t}\right)$.

6.3.3.

1. $1+e^t+\frac{1}{2}t^2$;
2. $t^3+e^{-t}\sin x$;
3. $(1+t)e^{-t}\cos x$;
4. $\operatorname{ch} t \sin x$;
5. $1-\cos t+(1+4t)^{-1/2}\exp\left(-\frac{x^2}{1+4t}\right)$.

6.3.4. Apply the method of separation of variables to the equation $u_t = a^2u_{xx}$ with the boundary conditions $u(0,t) = u(l,t) = 0$ and with the given initial conditions. The corresponding Sturm-Liouville problem is solved in Problem 6.2.11:1.

6.3.5. Solve the equation $u_t = a^2u_{xx}$ with the boundary conditions $u_x(0,t) = u_x(l,t) = 0$ and with the given initial conditions.

6.3.7. Solve the equation $u_t = a^2u_{xx}$ with the given conditions.

6.3.8. Solve the equation $u_t = a^2u_{xx} - b^2u$ with the given conditions.

In Problems 6.4.2: 2-4 and 6.4.3: 2-3 the Laplace equation is reduced to an ordinary differential equation, if we use the cartesian coordinates in 6.4.2: 1-2 and 6.4.3: 1, the polar coordinates in 6.4.2: 3-4, and the spherical coordinates in 6.4.3: 2-3.

6.4.2.

1. $u(x_1,x_2) = a(x_1^2 - x_2^2) + bx_1x_2 + cx_1 + dx_2 + e$, where a,b,c,d,e are constants.
2. $u(x) = ax_1 + b$;
3. $u(\rho) = a\ln\frac{1}{\rho} + b$ or $u(x_1,x_2) = a\ln\frac{1}{(x_1^2+x_2^2)^{1/2}} + b$;
4. $u(\varphi) = a\varphi + b$ or $u(x_1,x_2) = a\arctan\frac{x_2}{x_1} + b$.

6.4.3.

1. $u(x_1,x_2,x_3) = ax_1^2 + bx_2^2 + cx_3^2 + p(x_1,x_2,x_3)$, where a,b,c are constants such that $a+b+c=0$, and $p(x_1,x_2,x_3)$ is a linear combination of $x_1x_2, x_1x_3, x_1, x_2, x_3, 1$;
2. $u(r) = \frac{a}{r} + b$ or $u(x_1,x_2,x_3) = \frac{a}{(x_1^2+x_2^2+x_3^2)^{1/2}} + b$;
3. $u(\theta) = a\ln\tan\frac{\theta}{2} + b$ or

$$u(x_1,x_2,x_3) = a\ln\tan\left(\frac{(x_1^2+x_2^2)^{1/2}}{x_3} + (x_1^2+x_2^2+x_3^2)^{1/2}\right) + b.$$

6.4.4.

1. $k = -3$;
2. $k = \pm 2i$, and $\mathrm{ch}\, ky = \cos 2y$;
3. $k = \pm 3$.

6.4.5.

1. yes;
2. no;
3. yes.

6.4.6.

1. yes;
2. yes;
3. no.

6.4.8. Use the Cauchy-Riemann differential equations.

6.4.9. Calculate $Re\dfrac{1}{f(z)}$, where $f(z)$ is defined in 6.4.8.

6.4.10.

1. $f(z) = x_1^3 - 3x_1x_2^2 + i(3x_1^2x_2 - x_2^3) + i(-3y_1^2y_2 + y_2^3 + C)$;

3. $f(z) = \sin x_1 \operatorname{ch} x_2 + i\cos x_1 \operatorname{sh} x_2 + i(-\cos y_1 \operatorname{sh} y_2 + C)$;

4. $f(z) = z^2$;

5. $f(z) = \dfrac{z^2}{2i} + (1-i)z + iC$.

For solving Problems 4 and 5 the Goursat formula (6.4.2) is used.

6.4.12.

1. $2\pi u(0,0)$, use Problem 6.4.10: 4;

2. $2\pi u(0,0)$, use Problem 6.4.10: 3.

6.4.13.

1) $u(x_1,x_2) = a$;

2) $u(x_1,x_2) = ax_1 + bx_2$;

3) $u(x_1,x_2) = x_1x_2$;

5) $u(x_1,x_2) = \dfrac{a+b}{2} + \dfrac{b-a}{2}(x_1^2 - x_2^2)$ or

$$u(\rho,\varphi) = \frac{a}{2}\left(1 - \frac{\rho^2}{l^2}\cos 2\varphi\right) + \frac{b}{2}\left(1 + \frac{\rho^2}{l^2}\cos 2\varphi\right).$$

6.4.14.

1) $u(\rho,\varphi) = a$;

3) $u(\rho,\varphi) = \dfrac{1}{2}\dfrac{l^4}{\rho^2}\sin 2\varphi$;

4) $u(\rho,\varphi) = a + \dfrac{bl}{\rho}\sin\varphi$;

5) $u(\rho,\varphi) = \dfrac{a+b}{2} - \dfrac{a-b}{2}\dfrac{l^2}{\rho^2}\cos 2\varphi$.

6.4.15. $u(\rho) = a + (b-a)\dfrac{\ln \rho / l_1}{\ln \rho / l_2}$. The function u depends only on ρ since the boundary condition does not depend on φ.

6.4.16. $u(\rho) = \dfrac{1}{4}(\rho^2 - l^2)$.

6.4.17. $u(x_1, x_2, x_3) = a$.

6.4.18. $u(x_1, x_2, x_3) = \dfrac{la}{\rho}$.

6.4.19.

1) The problem has no solutions;

3) $u(x_1, x_2) = \dfrac{al}{2}(x_1^2 - x_2^2) + C$ or $u(\rho, \varphi) = \dfrac{al}{2}\rho^2 \cos 2\varphi + C$;

4) The problem has no solutions;

5) $u(\rho, \varphi) = \left(a + \dfrac{3}{4}b\right)\sin\varphi - \dfrac{b}{12l^2}\sin 3\varphi + C$ or

$$u(x_1, x_2) = \left(a + \frac{3}{4}b\right)x_2 - \frac{b}{12l^2}(3(x_1^2 + x_2^2)x_2 - 4x_2^3) + C.$$

6.4.20.

2) $G(x,y) = \dfrac{1}{2\pi}\ln\dfrac{1}{|x-y|} - \dfrac{1}{2\pi}\ln\dfrac{R}{|x|}\cdot\dfrac{1}{|x-y^1|}, \quad y^1 = y\dfrac{R^2}{|y|^2}$.

6.4.21.

1) $G(x,y) = \dfrac{1}{4\pi}\dfrac{1}{|x-y|} - \dfrac{1}{4\pi}\dfrac{1}{|x-y^1|}, \quad y^1 = (y_1, y_2, 2l - y_3)$.

6.4.22.

1) $G(x,y) = \dfrac{1}{4\pi}\displaystyle\sum_{n,k=0}^{1}\dfrac{(-1)^{k+n}}{|x-y_{n,k}|}$,

$$y_{00} = y = (y_1, y_2, y_3),\ y_{n,k} = (y_1, (-1)^n y_2, (-1)^k y_3);$$

2) $G(x,y) = \dfrac{1}{4\pi}\displaystyle\sum_{k=0}^{n-1}\left(\dfrac{1}{|x-y_k|} - \dfrac{1}{|x-\tilde{y}_k|}\right), \quad x = (\rho, \varphi, z)$,

$$y_0 = y = (\mu, \psi, z),\ y_k = \left(\mu, \frac{2\pi k}{n} + \psi, z\right),$$

$$\tilde{y}_k = \left(\mu, \frac{2\pi k}{n} - \psi, z\right);$$

3) $G(x,y) = \dfrac{1}{4\pi}\displaystyle\sum_{m,n,k=0}^{1}(-1)^{m+n+k}\dfrac{1}{|x-y_{m,n,k}|}$,

$$y_{m,n,k} = ((-1)^m y_1, (-1)^n y_2, (-1)^k y_3);$$

4) $G(x,y) = \dfrac{1}{4\pi}\sum_{k=0}^{1}\left(\dfrac{1}{|x-\tilde{y}_k|} - \dfrac{R}{|\tilde{y}_k||x-y'_k|}\right)$,

$$\tilde{y}_0 = y = (y_1,y_2,y_3),\ \tilde{y}_1 = (y_1,y_2,y_3),\ y'_k = \tilde{y}_k\frac{R^2}{|\tilde{y}_k|^2};$$

5) $G(x,y) = \dfrac{1}{4\pi}\sum_{n,k=0}^{1}(-1)^{n+k}\left(\dfrac{1}{|x-y_{n,k}|} - \dfrac{R}{|y_{n,k}||x-y'_{n,k}|}\right)$,

$$y_{00} = y = (y_1,y_2,y_3),\ y_{n,k} = (y_1,(-1)^n y_2,(-1)^k y_3),$$

$$y'_{n,k} = y_{n,k}\frac{R^2}{|y_{n,k}|^2};$$

7) for the construction of the Green's function make the reflections with respect to the planes $x_3 = 0$ and $x_3 = l$;
$G(x,y) = \dfrac{1}{4\pi}\sum_{n=-\infty}^{\infty}\left(\dfrac{1}{|x-y_n|} - \dfrac{1}{|x-\tilde{y}_n|}\right)$,

$$y_n = (y_1,y_2,2nl+y_3),\ \tilde{y}_n = (y_1,y_2,2nl-y_3);$$

8) $G(x,y) = \dfrac{1}{4\pi}\sum_{n=-\infty}^{\infty}\sum_{k=0}^{1}\left(\dfrac{1}{|x-y_{n,k}|} - \dfrac{1}{|x-\tilde{y}_{n,k}|}\right)$,

$$y_{n,k} = ((-1)^k y_1,y_2,2nl+y_3),\ \tilde{y}_{n,k} = ((-1)^k y_1,y_2,2nl-y_3).$$

6.4.23.

1) $G(x,y) = \dfrac{1}{2\pi}\sum_{n,k=0}^{1}(-1)^{n+k}\ln\dfrac{1}{|x-y_{n,k}|}$,

$$y_{n,k} = ((-1)^n y_1,(-1)^k y_2);$$

3) $G(x,y) = \dfrac{1}{2\pi}\sum_{k=0}^{1}\left(\ln\dfrac{1}{|x-y_k|} - \ln\dfrac{R}{|y_n||x-y'_k|}\right)$,

$$y_k = (y_1,(-1)^k y_2),\ y'_k = y_k\frac{R^2}{|y_k|};$$

4) $G(x,y) = \dfrac{1}{2\pi}\sum_{n,k=0}^{1}(-1)^{n+k}\left(\ln\dfrac{1}{|x-y_{n,k}|} - \ln\dfrac{R}{|y_{n,k}||x-y_{n,k}|}\right)$,

$$y_{n,k} = ((-1)^n y_1,(-1)^k y_2),\ y'_{n,k} = y_{n,k}\frac{R^2}{|y_{n,k}|^2};$$

5) $G(x,y)=\dfrac{1}{2\pi}\sum_{n=-\infty}^{\infty}\Big(\ln\dfrac{1}{|x-y_n|}-\ln\dfrac{1}{|x-\tilde{y}_n|}\Big)$,

$$y_n=(y_1,2\pi n+y_2),\ \tilde{y}_n=(y_1,2\pi n-y_2);$$

6) $G(x,y)=\dfrac{1}{2\pi}\sum_{n=-\infty}^{\infty}\sum_{k=0}^{1}\Big(\ln\dfrac{1}{|x-y_{n,k}|}-\ln\dfrac{1}{|x-\tilde{y}_{n,k}|}\Big)$,

$$y_{n,k}=((-1)^k y_1,2\pi n+y_2),\ \tilde{y}_{n,k}=((-1)^k y_1,2\pi n-y_2).$$

6.4.24.

1) $G(z,\zeta)=\dfrac{1}{2\pi}\ln\dfrac{|z-\overline{\zeta}|}{|z-\zeta|}$;

2) $G(z,\zeta)=\dfrac{1}{2\pi}\ln\dfrac{|z^2-\overline{\zeta}^2|}{|z^2-\zeta^2|}$,
the function $\omega(z,\zeta)$ is a superposition of the function z^2 and a linear-fractional function;

3) $G(z,\zeta)=\dfrac{1}{2\pi}\ln\dfrac{|z^n-\overline{\zeta}^n|}{|z^n-\zeta^n|}$;

4) $G(z,\zeta)=\dfrac{1}{2\pi}\ln\dfrac{|z-\overline{\zeta}||R^2-z\overline{\zeta}|}{|z-\zeta||R^2-z\zeta|}$,
the function $\omega(z,\zeta)$ is a combination of the functions: $z_1=-\dfrac{z}{R}$, $z_2=\dfrac{1}{2}(z+\dfrac{1}{z_1})$,
$z_3=e^{i\alpha}\dfrac{z_2-\zeta_2}{z_2-\overline{\zeta}_2}$;

5) $G(z,\zeta)=\dfrac{1}{2\pi}\ln\dfrac{|z^2-\overline{\zeta}^2||R^4-(z\overline{\zeta})^2|}{|z^2-\zeta^2||R^4-(z\zeta)^2|}$;

7) $G(z,\zeta)=\dfrac{1}{2\pi}\ln\dfrac{|\operatorname{ch}z-\overline{\operatorname{ch}\zeta}|}{|\operatorname{ch}z-\operatorname{ch}\zeta|}$.

6.4.28.

1. $u(x_1,x_2,x_3)=e^{-\sqrt{2}x_3}\cos x_1\cos x_2$;
2. $u(x_1,x_2,x_3)=(e^{-\sqrt{x_3}}-e^{\sqrt{x_3}})\cos x_1\cos x_2$;

6.4.29.

1. $u(x_1,x_2)=\dfrac{x_1}{x_1^2+(x_2+1)^2}$;
2. $u(x_1,x_2)=\dfrac{\cos x_1\operatorname{sh}(\pi-x_2)}{\operatorname{sh}\pi}$.

6.4.30. No.

6.4.31. $f(x_1,x_2,x_3) = -20(x_1^2+x_2^2+x_3^2)$.

6.4.32. Show that in polar coordinates the function $V(x_1,x_2)$ is a solution of the problem

$$\Delta V(\rho) = \begin{cases} 1, & \rho \le 1, \\ 0, & \rho > 1, \end{cases} \qquad V(1+0) = V(1-0),\ V_\rho(1+0) = V_\rho(1-0).$$

6.4.33. Show that in spherical coordinates the function $V(x_1,x_2,x_3)$ is a solution of the problem

$$\Delta V(\rho) = \begin{cases} -1, & \rho \le 1, \\ 0, & \rho > 1, \end{cases} \qquad V(1-0) = V(1+0),\ V_\rho(1-0) = V_\rho(1+0).$$

6.4.34. $-\dfrac{16}{3}$.

6.4.35. $f(x_1,x_2) = 2\pi x_1$, $V(x_1,x_2) = \dfrac{\pi x_1}{4|x|^2}$. For solving the exterior Dirichlet problem introduce polar coordinates and then apply the method of separation of variables.

6.4.36.

$$V(x) = \begin{cases} \dfrac{f_0 R^3}{3|x|}, & |x| \ge R, \\ \dfrac{1}{2} f_0 \left(R^2 - \dfrac{|x|^2}{3} \right), & |x| < R; \end{cases}$$

1. In spherical coordinates the problem is reduced to calculating the integral

$$\int_0^R \int_0^\pi \frac{\zeta^2 \sin\theta\, d\theta d\zeta}{\sqrt{\zeta^2+\rho^2-2\zeta\rho\cos\theta}},$$

which can be computed by the change of the variable
$\psi = \sqrt{\zeta^2+\rho^2-2\zeta\rho\cos\theta}$; (instead of θ).

2. Using the spherical symmetry of the potential, write the Laplace operator in the form

$$\Delta V(\rho) = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left(\rho^2 \frac{\partial V}{\partial \rho} \right).$$

Integrating the corresponding equations inside and outside the domain one gets

$$V(\rho) = \begin{cases} \dfrac{C_1}{\rho} + C_2, & \rho \ge R, \\ -\dfrac{f_0}{6}\rho^2 + C_3, & \rho < R. \end{cases}$$

The constants C_i $(i = \overline{1,3})$ are found by using the continuity of the function $V(\rho)$ and its normal derivatives and $\lim\limits_{\rho\to\infty} V(\rho) = 0$.

6.4.37.

1. $$V(x)=\begin{cases}-\frac{1}{2}f_0(b^2-a^2), & |x|<a,\\ \frac{1}{2}f_0b^2-\frac{f_0}{6}\left(|x|+\frac{2a^3}{|x|}\right), & a<|x|<b;\\ \frac{1}{3}f_0(b^3-a^3)\frac{1}{|x|}, & |x|>b.\end{cases}$$

2. $$V(x)=\begin{cases}\frac{1}{2}\left[f_1\left(a^2+\frac{|x|^2}{3}\right)+f_2(c^2-b^2)\right], & |x|<a,\\ \frac{1}{2}f_2(c^2-b^2)+\frac{f_1}{3}a^3\frac{1}{|x|}), & a<|x|<b;\\ \frac{f_2(c^3-b^3)+a^3f_1}{3|x|}, & |x|>c.\end{cases}$$

6.4.39.

1. $W(\rho,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{(\rho^2-r^2)g(\zeta)\,d\zeta}{r^2+\rho^2-2r\rho\cos(\zeta-\theta)}$, where ρ,θ are polar coordinates;

2. $W(x_1,x_2)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x_2g(\zeta)\,d\zeta}{(x_1-3)^2+x_2^2}$;

3. $W(x_1,x_2,x_3)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{x_3g(\zeta,\eta)\,d\zeta d\eta}{[(x_1-\zeta)^2+(x_2-\eta)^2+x_3^2]^{3/2}}$.

6.4.40.

1. $$W(x)=\begin{cases}\nu_0, & |x|<1,\\ \frac{1}{2}\nu_0, & |x|=1,\\ 0, & |x|>1.\end{cases}$$

It follows from (6.4.8) that $W^+(x)=\nu_0$, $W^-(x)=0$. The solution of the interior Dirichlet problem

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial W}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2W}{\partial\theta^2}=0,$$
$$W|_{\rho=1}=\nu_0$$

(ρ,θ are polar coordinates) is found with the help of the method of separation of variables:

2. $$W(x_1,x_2)=\begin{cases}\frac{x_1}{2}, & |x|<1,\\ 0, & |x|=1,\\ -\frac{x_1}{2|x|^2}, & |x|>1.\end{cases}$$

Taking (6.4.8) into account solve the interior and exterior Dirichlet problems:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial W}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2W}{\partial\theta^2}=0,\quad W|_{\rho=1}=\frac{1}{2}\cos\theta;$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial W}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 W}{\partial\theta^2}=0, \quad W|_{\rho=1}=-\frac{1}{2}\cos\theta,$$

by the method of separation of variables.

6.4.42.

$$u(x_1,x_2,x_3)=\begin{cases}\mu_0, & |x|\le 1,\\ \dfrac{\mu_0}{|x|}, & |x|>1;\end{cases}$$

1. Introduce spherical coordinates and instead of θ (θ has the range $0\le\theta\le\pi$) use a new variable $\psi=\sqrt{\rho^2+1-2\rho\cos\theta}$.
2. Using the spherical symmetry of the potential write the Laplace equation in the form

$$\frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial u}{\partial\rho}\right)=0.$$

For finding the constants use (6.4.20), (6.4.21), the continuity properties for $u(\rho)$ and

$$\lim_{\rho\to\infty}u(\rho)=0.$$

6.4.43.

$$u(x)=\begin{cases}-R\ln R, & |x|\le R,\\ -R\ln|x|, & |x|>R.\end{cases}$$

The problem is reduced to finding the constants appearing in the integration of the equation

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right)=0.$$

6.4.44. $u(x_1,x_2)=-\frac{1}{2}x_2$. Introduce polar coordinates and use the method of separation of variables.

6.4.45. $\mu(x_1,x_2)=2x_1+8x_1x_2$. Use the relations (4.17)-(4.18).

6.4.46.

$$u(\rho,\varphi)=\frac{r}{\pi}\int_0^{2\pi}g(\psi)\ln\frac{1}{r^2+\rho^2-2r\rho\cos(\theta-\psi)}d\psi+C.$$

Determine the solution of the problem

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial\theta^2}=0, \quad \left.\frac{\partial u}{\partial\rho}\right|_{\rho=r}=g(\theta)$$

in the form

$$u(\rho,\varphi)=\frac{r}{2\pi}\int_0^{2\pi}\mu(\psi)\ln\frac{1}{r^2+\rho^2-2r\rho\cos(\theta-\psi)}d\psi+C,$$

hence

$$\frac{\partial u}{\partial \rho}\Big|_{\rho=r} = -\frac{1}{4\pi}\int_0^{2\pi} \mu(\tau)\, d\tau.$$

Using (6.4.17) and the condition $\int_0^{2\pi} f(\tau)\, d\tau = 0$ for the solvability of the interior Neumann problem one has $\mu(\tau) = 2g(\tau)$.

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